JOHN ADAMSKI
2/13/2017

§13.3 FUNDAMENTAL THM FOR LINE INTEGRALS

RECALL F.T.C. PART II (EVALUATION THM) FOR FUNCTION f OF 1 VARIABLE INTEGRATED OVER AN INTERVAL $I = [a, b]$

$$\int_I f'(x)\,dx = \int_a^b f'(x)\,dx = f(b) - f(a)$$

SCALAR MULTIPLICATION

DERIVATIVE OF f

NOW, IN HIGHER DIMENSIONS, f IS A FUNCTION OF SEVERAL VARIABLES INTEGRATED OVER A CURVE $C = \vec{r}(t),\ a \le t \le b$.

$$\boxed{\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))}$$

PROOF:

$$\int_C \nabla f \cdot d\vec{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle dt$$

$$= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{d}{dt}\left[f(\vec{r}(t)) \right] dt \underset{\text{FTC. II}}{=} f(\vec{r}(b)) - f(\vec{r}(a)) \qquad \square$$

ex. FIND WORK DONE BY FORCE FIELD $\vec{F}(\vec{r}) = \dfrac{k\vec{r}}{|\vec{r}|^3}$, K CONSTANT,

AS PARTICLE MOVES ALONG STRAIGHT LINE FROM $(2,0,0)$ TO $(2,1,5)$.

OBSERVE THAT FOR $f(x,y,z) = \dfrac{-K}{\sqrt{x^2+y^2+z^2}}$ WE HAVE

$$\nabla f = \frac{-K}{\sqrt{x^2+y^2+z^2}^{\,3}} \langle x, y, z \rangle = \frac{-K\vec{r}}{|\vec{r}|^3}$$

THUS, $W = \int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$

$$= f(\vec{r}(b)) - f(\vec{r}(a))$$

$$= f(2,1,5) - f(2,0,0)$$

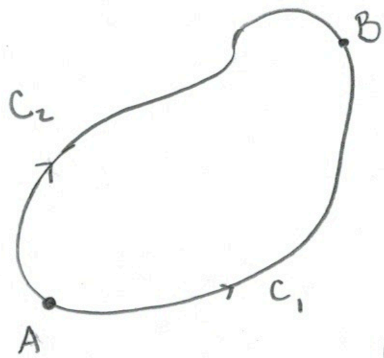
$$= \frac{-K}{\sqrt{2^2+1^2+5^2}} + \frac{K}{\sqrt{4+0+0}} = K\left(\frac{1}{2} - \frac{1}{\sqrt{30}}\right)$$

NOTE THAT WE DID NOT HAVE TO PARAMETERIZE THE CURVE!

WHEN OUR VECTOR FIELD IS A GRADIENT VECTOR FIELD (CONSERVATIVE) LINE INTEGRALS ARE PATH INDEPENDENT: ANY SMOOTH CURVE WITH SAME BEGINNING & END POINTS YIELDS SAME LINE INTEGRAL.

A CURVE IS CALLED <u>CLOSED</u> IF ITS TERMINAL POINT = INITIAL POINT.

IF $\vec{F}$ IS $\int_C \vec{F}\cdot d\vec{r}$ IS PATH IND. $\forall C$ & C IS CLOSED, WHAT IS $\int_C \vec{F}\cdot d\vec{r}$?



SUPPOSE A = INIT. PNT. = TERM. PNT OF C.

CHOOSE ANOTHER POINT B ON C, ARBITRARY.

CALL THE 2 CURVES CONNECTING A AND B

C_1 & C_2. WE KNOW $\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$.

FURTHERMORE $\int_{-C_2} \vec{F}\cdot dr = -\int_{C_2} \vec{F}\cdot d\vec{r}$, SO

$$\int_C \vec{F}\cdot dr = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = 0.$$

IF $\int_C \vec{F}\cdot d\vec{r} = 0$ WHENEVER C IS CLOSED, THEN TAKE A, B

ARBITRARY PNTS & LET C_1, C_2 BE ANY TWO PATHS CONNECTING

A & B, AND $C = C_1 \oplus -C_2$ (AS ABOVE) SO C IS CLOSED.

$$0 = \int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{-C_2} \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot dr$$

$$\Rightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$$

IN SUMMARY,

THM: $\int_C \vec{F} \cdot d\vec{r}$ IS INDEPENDENT OF PATH IN D (DOMAIN)

IFF $\int_C \vec{F} \cdot d\vec{r} = 0$ FOR EVERY CLOSED PATH C IN D.

AND, THE ONLY VECTOR FIELD THAT ARE INDEPENDENT OF PATH ARE CONSERVATIVE (GRADIENT VECTOR FIELDS.)

THM: SUPPOSE $\vec{F}$ IS A VECTOR FIELD THAT IS CONTINUOUS ON AN OPEN CONNECTED REGION D. IF $\int_C \vec{F} \cdot d\vec{r}$ IS IND. OF PATH IN D, THEN $\vec{F}$ IS A CONSERVATIVE VECTOR FIELD IN D; THAT IS, $\exists f$ S.T. $\nabla f = \vec{F}$.

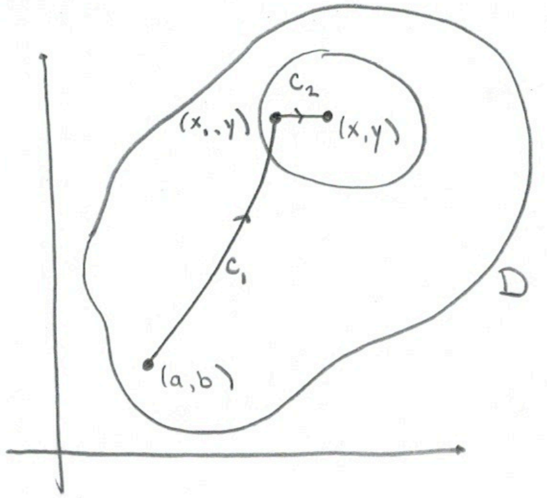
PROOF: LET $A(a,b)$ BE FIXED PNT IN D.

WE CONSTRUCT POTENTIAL FUNCTION f S.T. $\nabla f = \vec{F}$ BY

SETTING
$$f(x,y) = \int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r}$$

(PATH IND $\Rightarrow$ PATH FROM (a,b) TO (x,y) IS INCONSEQUENTIAL)

SINCE D IS OPEN $\exists$ DISK WITH CENTER (x,y), CONTAINED IN D



CHOOSE (x_1, y) IN THE DISK WITH $x_1 < x$

& LET $C = C_1 \oplus C_2$ (SEE FIGURE)

THEN

$$f(x,y) = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = \underbrace{\int_{(a,b)}^{(x_1,y)} \vec{F} \cdot d\vec{r}}_{\text{DOES NOT DEPEND ON } x} + \int_{(x_1,y)}^{(x,y)} \vec{F} \cdot d\vec{r}$$

$$\Rightarrow \frac{\partial}{\partial x} f(x,y) = 0 + \frac{\partial}{\partial x} \int_{C_2} \vec{F} \cdot d\vec{r}$$

WRITE $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$. THEN

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int_{C_2} P(x,y)\,dx + Q(x,y)\,dy$$

ON C_2, y IS CONSTANT SO $dy = 0$

$x = t$, $x_1 \leq t \leq x$

$dx = dt$

$$\Rightarrow \frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} \int_{x_1}^{x} P(t,y)\,dt = P(x,y) \quad \text{(F.T.C. PART 1)}$$

SIMILAR ARGUMENT USING VERTICLE SEGMENT SHOWS THAT

$$\frac{\partial}{\partial y} f(x,y) = Q(x,y)$$

THUS, $\vec{F} = \langle P, Q \rangle = \langle f_x, f_y \rangle = \nabla f$ (CONSERVATIVE) $\square$

HOW DO WE KNOW WHEN A VECTOR FIELD IS CONSERVATIVE?

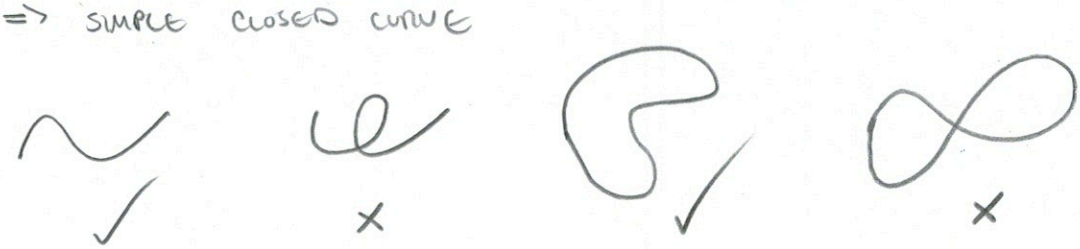
> <u>THM</u>: IF $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$ IS A CONSERVATIVE VECTOR FIELD, WHERE P & Q HAVE CONTINUOUS 1^{st} ORDER PARTIAL DERIVATIVES ON D, THEN THROUGHOUT D WE HAVE
>
> $$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

NOTE: THIS SAYS $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ [CLAIRAUT'S THM]

<u>ex.</u> $\vec{F}(x,y) = \langle x^2 + xy, y^2 - xy \rangle$ NOT CONSERVATIVE.

THE CONVERSE STATEMENT REQUIRES SOME DEFINITIONS.

<u>Def</u>: A SIMPLE CURVE IS A CURVE THAT DOES NOT INTERSECT ITSELF ANYWHERE BETWEEN ITS ENDPOINTS. IF ENDPNTS INTERSECT $\Rightarrow$ SIMPLE CLOSED CURVE



Def: A SIMPLY CONNECTED REGION IN THE PLANE IS A CONNECTED REGION D SUCH THAT EVERY SIMPLE CLOSED CURVE IN D ENCLOSES ONLY PNTS IN D.

(i.e. ONE PIECE WITH NO HOLES)

THM LET $\vec{F} = \langle P, Q \rangle$ BE VECTOR FIELD ON OPEN SIMPLY-CONNECTED REGION D. SUPPOSE P, Q HAVE CONTINUOUS 1st ORDER PARTIAL DERIVATIVES AND

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{THROUGHOUT } D.$$

THEN $\vec{F}$ IS CONSERVATIVE.

(PROOF: DELAYED TILL §13.4)

ex. DETERMINE WHETHER OR NOT $\vec{F}(x,y) = \langle 3+2xy, x^2-3y^2 \rangle$ IS CONSERVATIVE.

(YES, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = 2x$ & $D = \mathbb{R}^2$ SIMP. CONN.

CONTINUOUS ON $\mathbb{R}^2$)

ex. FIND f S.T. $\vec{F} = \nabla f$.

ex. IF $\vec{F}(x,y,z) = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$, FIND f S.T. $\nabla f = \vec{F}$.

READ ON YOUR OWN LAST SECTION: "CONSERVATION OF ENERGY" p.785-6.