§ 13.3 THE FUNDAMENTAL THEOREM FOR LINE INTEGRALS

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1. $\int_C \nabla f \cdot d\vec{r} = f(\text{TERMINAL POINT}) - f(\text{INITIAL POINT})$

$= 50 - 10 = \boxed{40}$

2. WRITE $\vec{r}(t) = \langle t^2+1, t^3+t \rangle$, $0 \le t \le 1$

THEN $\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$= f(2,2) - f(1,0) = 9 - 3 = \boxed{6}$

3. (1) $\vec{F}$ HAS DOMAIN $\mathbb{R}^2$, WHICH IS SIMPLY CONNECTED.

(2) $P(x,y) = 2x - 3y$ AND $Q(x,y) = -3x + 4y - 8$

HAVE PARTIAL DERIVATIVES THAT ARE CONTINUOUS (IN FACT, CONSTANT) ON THE DOMAIN, $\mathbb{R}^2$.

(3) $\dfrac{\partial P}{\partial y} = -3 = \dfrac{\partial Q}{\partial x}$ THROUGHOUT DOMAIN $\mathbb{R}^2$.

$\therefore$ $\vec{F}$ IS CONSERVATIVE (BY THM 6)

TO FIND f: $\nabla f = \vec{F}$

$\dfrac{\partial f}{\partial x} = P = 2x - 3y \Rightarrow f(x,y) = x^2 - 3xy + g(y)$

THEN $\dfrac{\partial f}{\partial y} = -3x + g'(y) = Q = -3x + 4y - 8$

$\Rightarrow g'(y) = 4y - 8 \Rightarrow g(y) = 2y^2 - 8y + C$

$$\therefore \boxed{f(x,y) = x^2 - 3xy + 2y^2 - 8y + c}$$

5. $\vec{F}$ HAS SIMPLY CONNECTED DOMAIN ($\mathbb{R}^2$)

AND P, Q HAVE CONTINUOUS PARTIAL DERIVATIVES THROUGHOUT DOMAIN,

BUT

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left[e^x \cos y\right] = -e^x \sin y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[e^x \sin y\right] = e^x \sin y$$

NOT THE SAME, SO $\vec{F}$ IS $\boxed{\text{NOT CONSERVATIVE}}$

7. DOMAIN $D = \mathbb{R}^2$ IS SIMPLY CONNECTED

P, Q HAVE CONTINUOUS PARTIAL DERIVATIVES ON D

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left[ye^x + \sin y\right] = e^x + \cos y$$

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left[e^x + x\cos y\right] = e^x + \cos y$$

SAME

$\therefore \vec{F}$ IS CONSERVATIVE.

$$\frac{\partial f}{\partial x} = P = ye^x + \sin y \Rightarrow f = ye^x + x\sin y + g(y)$$

$$\frac{\partial f}{\partial y} = e^x + x\cos y + g'(y) = Q = e^x + x\cos y \Rightarrow g'(y) = 0$$

$$\Rightarrow g(y) = c \quad \text{SO} \quad \boxed{f(x,y) = ye^x + x\sin y + c}$$

9. DOM $(\vec{F}) = \{ (x,y) \mid \underbrace{y > 0}_{\text{OTHERWISE } \ln y \text{ UNDEFINED}} \}$ = "UPPER HALF-PLANE"

SO DOM $(\vec{F})$ IS SIMPLY CONNECTED, AND P, Q HAVE CONTINUOUS PARTIAL DERIVATIVES THROUGHOUT DOM $(\vec{F})$. SO $\vec{F}$ IS CONSERVATIVE.

$$\frac{\partial f}{\partial x} = P = \ln y + 2xy^3 \Rightarrow f = x\ln y + x^2y^3 + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{x}{y} + 3x^2y^2 + g'(y) = Q = 3x^2y^2 + \frac{x}{y} \Rightarrow g'(y) = 0$$

SO $g(y) = c$ AND $\boxed{f(x,y) = x\ln y + x^2y^3 + c}$

11. (a) $\vec{F}(x,y) = \langle xy^2, x^2y \rangle$

$f_x = xy^2 \Rightarrow f = \frac{1}{2}x^2y^2 + g(y)$

$f_y = x^2y + g'(y) = x^2y \Rightarrow g'(y) = 0 \Rightarrow g(y) = c$

$\therefore \boxed{f(x,y) = \frac{1}{2}x^2y^2 + c}$ (NOTE THAT WE MAY SET $c = 0$)

(b) $$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\overset{b}{1})) - f(\vec{r}(\overset{a}{0}))$$

$$= f\left(1 + \sin\frac{\pi}{2}, 1 + \cos\frac{\pi}{2}\right) - f(0 + \sin 0, 0 + \cos 0)$$

$$= f(2,1) - f(0,1)$$

$$= \frac{1}{2}(2)^2(1)^2 - \frac{1}{2}(0)^2(1)^2 = 2 - 0 = \boxed{2}$$

13. (a) $\vec{F}(x,y,z) = \langle yz, xz, xy+2z \rangle$

$f_x = yz \Rightarrow f = xyz + g(y,z)$

$\Rightarrow f_y = xz + g_y(y,z) = xz \Rightarrow g_y = 0 \Rightarrow g(y,z) = g(z)$

$\Rightarrow f_z = xy + g'(z) = xy + 2z \Rightarrow g'(z) = 2z \Rightarrow g(z) = z^2 + c$

$\therefore \boxed{f(x,y,z) = xyz + z^2}$ (LET $c = 0$).

(b) $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot dr = f(4,6,3) - f(1,0,-2)$

$= (4)(6)(3) + (3)^2 - \left[(1)(0)(-2) + (-2)^2\right] = 81 - 4 = \boxed{77}$

15. $\vec{F}(x,y,z) = \langle yze^{xz}, e^{xz}, xye^{xz} \rangle$

$f_x = yze^{xz} \Rightarrow f = ye^{xz} + g(y,z)$

THEN $f_y = e^{xz} + g_y(y,z) = e^{xz} \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z) = g(z)$

THEN $f_z = xye^{xz} + g'(z) = xye^{xz} \Rightarrow g'(z) = 0 \Rightarrow g(z) = c$

$\therefore$ LET $\boxed{f(x,y,z) = ye^{xz}}$

(b) $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(2)) - f(\vec{r}(0))$

$= f(5,3,0) - f(1,-1,0) = 3e^{5 \cdot 0} + e^{1 \cdot 0}$

$= 3 + 1 = \boxed{4}$

$$\qquad\qquad P \qquad\qquad\quad Q$$

17. $\vec{F}(x,y) = \langle 2xe^{-y}, \ 2y - x^2e^{-y} \rangle$

(1) $\text{DOM}(\vec{F}) = \mathbb{R}^2$ IS OPEN & SIMPLY CONNECTED

(2) $\dfrac{\partial P}{\partial y} = -2xe^{-y} = \dfrac{\partial Q}{\partial x}$ ON $\text{DOM}(\vec{F}) = \mathbb{R}^2$

$\therefore$ $\vec{F}$ IS CONSERVATIVE, HENCE $\int_C \vec{F} \cdot d\vec{r}$ IS IND. OF PATH.

FIND f S.T. $\vec{F} = \nabla f$:

$f_x = P = 2xe^{-y} \Rightarrow f = x^2e^{-y} + g(y)$

$\Rightarrow f_y = -x^2e^{-y} + g'(y) = 2y - x^2e^{-y} \Rightarrow g'(y) = 2y \Rightarrow g(y) = y^2 + C$

$\therefore$ LET $f(x,y) = x^2e^{-y} + y^2$

THEN $\int_C \vec{F} \cdot dr = \int_C \nabla f \cdot d\vec{r} = f(2,1) - f(1,0)$

$= (2)^2e^{-(1)} + (1)^2 - \left[(1)^2e^{-(0)} + (0)^2\right] = \dfrac{4}{e} + 1 - 1 - 0 = \boxed{\dfrac{4}{e}}$

19. NOTE THAT $\text{DOM}(\vec{F}) = \{(x,y) \mid y \geq 0\} \supset \underbrace{\{(x,y) \mid y > 0\}}_{\text{OPEN, SIMPLY CONNECTED.}} = D$

ALSO, $\dfrac{\partial P}{\partial y} = 3\sqrt{y} = \dfrac{\partial Q}{\partial x}$ THROUGHOUT D. $\therefore$ $\vec{F}$ IS CONSERVATIVE ON D,

HENCE $\int_C \vec{F} \cdot d\vec{r}$ IS PATH INDEPENDENT.

FIND f S.T. $\vec{F} = \nabla f$:

$\vec{F}(x,y) = \langle 2y^{3/2}, 3x\sqrt{y} \rangle$

$f_x = 2y^{3/2} \Rightarrow f = 2xy^{3/2} + g(y)$

THEN $f_y = 3x\sqrt{y} + g'(y) = 3x\sqrt{y} \Rightarrow g'(y) = 0 \Rightarrow g(y) = c$

LET $f(x,y) = 2xy^{3/2}$

THEN $\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(2,4) - f(1,1)$

$= 2(2)(4)^{3/2} - 2(1)(1)^{3/2} = 32 - 2 = \boxed{30}$

21. $\boxed{\text{No.}}$ THE LINE INTEGRAL OF $\vec{F}$ OVER ANY CIRCLE CENTERED AT THE ORIGIN WITH RADIUS > 0 TRAVELLED COUNTER-CLOCKWISE WILL BE POSITIVE SINCE IN THIS CASE $\vec{F} \cdot \vec{r}\,'(t) > 0$ THROUGHOUT. IN ORDER TO BE CONSERVATIVE ALL LINE INTEGRALS OVER CLOSED PATHS MUST BE ZERO.

25. RECALL THAT IF $f(x,y,z)$ HAS CONTINUOUS 2^{nd} ORDER PARTIAL DERIVATIVES, THEN THE ORDER IN WHICH YOU TAKE MIXED 2^{nd} PARTIAL DERIVATIVES DOES NOT MATTER. THAT IS,

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial^2 f}{\partial z \partial x} = \frac{\partial^2 f}{\partial x \partial z}, \quad \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial^2 f}{\partial y \partial z}.$$

e.g. $\frac{\partial f_x}{\partial y}$

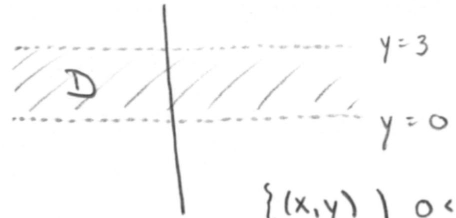
IF $\vec{F} = \langle P, Q, R \rangle$ IS CONSERVATIVE THEN $f_x = P$, $f_y = Q$, $f_z = R$.

$$\therefore \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}$$

27. (a) OPEN

(b) CONNECTED

(c) SIMPLY CONNECTED

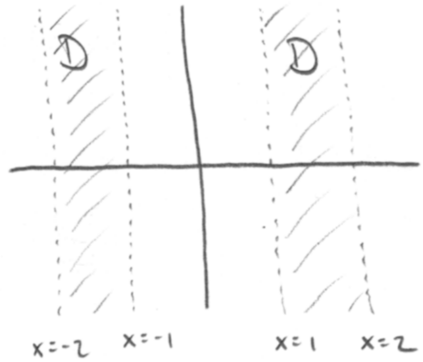


$\{(x,y) \mid 0 < y < 3\}$

28. (a) OPEN

(b) NOT CONNECTED

(c) NOT SIMPLY CONNECTED



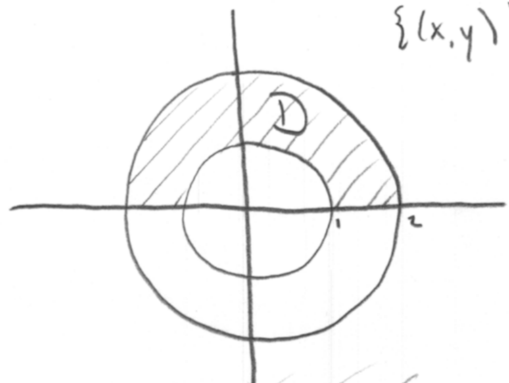
$\{(x,y) \mid 1 < |x| < 2\}$

29. (a) NOT OPEN

(b) CONNECTED

(c) SIMPLY CONNECTED

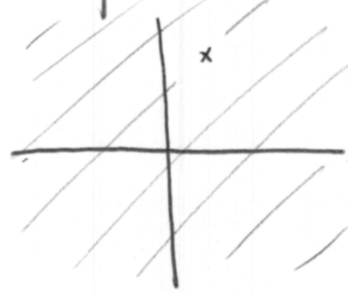
$\{(x,y) \mid 1 \le x^2 + y^2 \le 4$ AND $y \ge 0\}$



30. (a) OPEN

(b) CONNECTED

(c) NOT SIMPLY CONNECTED



$\{(x,y) \mid (x,y) \neq (2,3)\}$

31. (a) $\vec{F}(x,y) = \left\langle \dfrac{-y}{x^2+y^2}, \dfrac{x}{x^2+y^2} \right\rangle$

$$\left.\begin{aligned} \frac{\partial P}{\partial y} &= \frac{-(x^2+y^2)+y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial Q}{\partial x} &= \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}\right\} \text{SAME.}$$

(b) $\int_{C_1} \vec{F}\cdot d\vec{r}$, $\vec{r}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -\sin t, \cos t \rangle$

$$= \int_0^\pi \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^\pi \langle -\sin t, \cos t\rangle \cdot \langle -\sin t, \cos t \rangle\,dt$$

$$= \int_0^\pi \sin^2 t + \cos^2 t\,dt = \int_0^\pi dt = t\Big|_0^\pi = \pi$$

BUT $\int_{C_2} \vec{F}\cdot d\vec{r}$, $\vec{r}(t) = \langle \cos t, -\sin t \rangle$, $0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -\sin t, -\cos t \rangle$

$$= \int_0^\pi \langle \sin t, \cos t \rangle \cdot \langle -\sin t, -\cos t \rangle\,dt = -\int_0^\pi dt = -\pi$$

SO $\int_{C_1} \vec{F}\cdot d\vec{r} \neq \int_{C_2} \vec{F}\cdot d\vec{r}$. BUT THIS DOES NOT CONTRADICT THM 6

BECAUSE $\text{Dom}(\vec{F}) = \{(x,y) \mid (x,y) \neq (0,0)\}$ IS <u>NOT</u> SIMPLY CONNECTED (AND $C_1 + (^-C_2)$ ENCIRCLES THIS REMOVED POINT).