§13.4 GREEN'S THM

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LET C BE A $\underbrace{\text{POSITIVELY ORIENTED}}_{}$, PIECEWISE-SMOOTH, SIMPLE CLOSED CURVE IN THE PLANE & LET D BE THE REGION BOUNDED BY C (INCLUDING C).

"REGION ON LEFT"

IF P & Q HAVE CONTINUOUS PARTIAL DERIVATIVES ON AN OPEN REGION THAT CONTAINS D, THEN

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

THIS NOTATION JUST EMPHASIZES THAT C IS CLOSED & POSITIVELY ORIENTED.

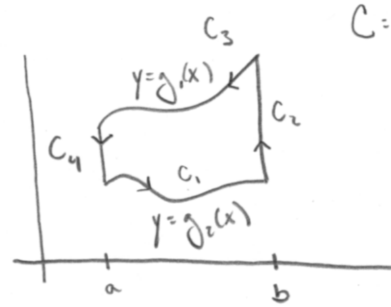
REMARK: COUNTERPART OF FTC FOR DOUBLE INTEGRALS.

$$F(b) - F(a) = \int_a^b F'(x)\,dx$$

$$\int_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

ex. EVALUATE $\int_C x^4\,dx + xy\,dy$ WHERE C IS THE TRIANGULAR CURVE CONSISTING OF LINE SEGMENTS FROM $(0,0)$ TO $(1,0)$, $(1,0)$ TO $(0,1)$ & $(0,1)$ TO $(0,0)$.

$C = C_1 + C_2 + C_3 + C_4$



PROOF OF GREEN'S THM

ASSUME D IS A TYPE I REGION:

$$D = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}.$$

WE SHOW $\oint_C P\,dx = -\iint_D \frac{\partial P}{\partial y}\,dA$.

WE HAVE

$$\iint_D \frac{\partial P}{\partial y}\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\,dy\,dx$$

$$= \int_a^b \Big(P(x, g_2(x)) - P(x, g_1(x))\Big)\,dx \qquad (1)$$

ON THE OTHER HAND,

$$\oint_C P\,dx = \int_{C_1} P\,dx + \underbrace{\int_{C_2} P\,dx}_{0} - \int_{-C_3} P\,dx + \underbrace{\int_{C_4} P\,dx}_{0}$$

x CONSTANT $\Rightarrow dx = 0$

$$= \int_a^b P(x, g_1(x))\,dx - \int_a^b P(x, g_2(x))\,dx \qquad (2)$$

(2) = −(1). DONE. $\left(\oint_C Q\,dy = \iint_D \frac{\partial Q}{\partial x}\,dA \right.$ IS PROVED SIMILARLY FOR TYPE II REGIONS $\left.\right)$ $\square$

ex. EVALUATE $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$

WHERE C IS CIRCLE $x^2 + y^2 = 9$. (36π)

(USE GEOMETRY!)

REVERSE DIRECTION : CALCULATE AREA

$$\text{AREA}(D) = \iint_D 1\,dA = \oint_C P\,dx + Q\,dy$$

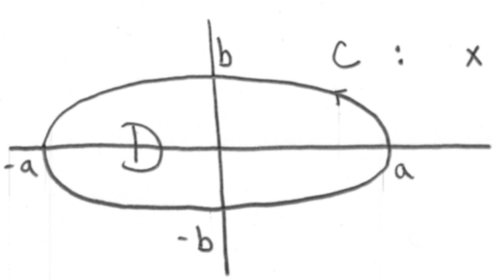
ANY $\vec{F} = \langle P, Q \rangle$ WITH $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1$

SUCH AS : (1) $\vec{F} = \langle 0, x \rangle$

(2) $\vec{F} = \langle -y, 0 \rangle$

(3) $\vec{F} = \frac{1}{2}\langle -y, x \rangle$

ex. FIND AREA OF ELLIPSE $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

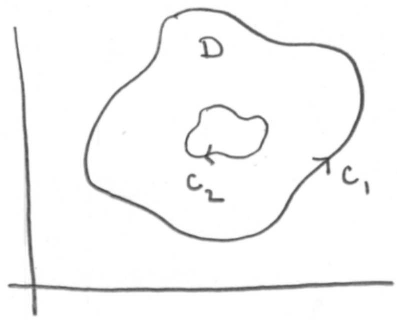


C : $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$

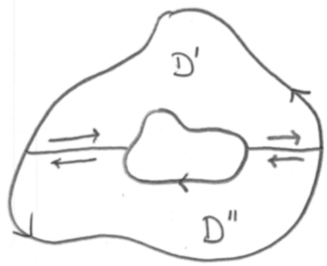
$$\iint_D 1\,dA = \int_0^{2\pi} \tfrac{1}{2}\langle -b\sin t, a\cos t\rangle \cdot \langle -a\sin t, b\cos t\rangle\,dt$$

$$= \int_0^{2\pi} \tfrac{1}{2}ab\,dt = \boxed{\pi ab}$$

EXTENSION TO REGIONS WITH HOLES: (NOT S.C.)



BOUNDARY: C_1 AND C_2 (POS. ORIENTED)



DIVIDE D INTO D' & D''. THEN

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_{D'} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D''} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_{\partial D'} P\,dx + Q\,dy + \int_{\partial D''} P\,dx + Q\,dy$$

NOTE THAT THE LINE INTEGRALS ALONG SHARED BOUNDARY CURVES ARE IN OPPOSITE DIRECTIONS.

THUS, THEY CANCEL.

$$\rightarrow \; = \int_{C_1} P\,dx + Q\,dy + \int_{C_2} P\,dx + Q\,dy = \int_C P\,dx + Q\,dy.$$

PROOF OF THM 13.3.6

LET $\vec{F} = \langle P, Q \rangle$ BE A VECTOR FIELD ON OPEN SIMPLY CONN. REGION D. SUPPOSE P, Q HAVE CONT. 1ST ORDER DERIV.'S &

$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ THROUGHOUT D. THEN $\vec{F}$ IS CONSERVATIVE.

Pf: LET C BE ANY SIMPLE CLOSED CURVE IN D, AND R IS REGION BOUNDED BY C.

BY GREEN'S THM,

$$\oint_C \vec{F} \cdot dr = \oint_C P\,dx + Q\,dy = \iint_R \underbrace{\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}}_{0}\, dA = 0.$$

CLOSED CURVES THAT ARE NOT SIMPLE, CAN BE BROKEN UP INTO CLOSED CURVES THAT ARE SIMPLE.

THUS $\oint_C \vec{F} \cdot d\vec{r} = 0$ FOR ANY CLOSED CURVE C

$\Rightarrow \int_C \vec{F} \cdot d\vec{r}$ PATH INDEPENDENT $\Rightarrow \vec{F}$ CONSERVATIVE. $\square$