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§13.4 GREEN'S THEOREM

1, 4, 5, 7, 9-12, 19, 21-23, 28

1. (a) $\oint_C (x-y)\,dx + (x+y)\,dy$

C: $x = 2\cos t \qquad y = 2\sin t \qquad 0 \le t \le 2\pi$

$dx = -2\sin t\,dt \qquad dy = 2\cos t\,dt$

$$\rightsquigarrow \int_0^{2\pi} \Big((2\cos t - 2\sin t)(-2\sin t) + (2\cos t + 2\sin t)\,2\cos t \Big)\,dt$$

$$= \int_0^{2\pi} \left(-4\sin t\cos t + 4\sin^2 t + 4\cos^2 t + 4\sin t\cos t \right) dt$$

$$= 4\int_0^{2\pi} dt = \boxed{8\pi}$$

(b) $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 1 - (-1) = 2$

SO WE HAVE $\displaystyle\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$= 2\iint_D dA$ WHERE D IS DISK OF RADIUS 2 WITH AREA $\pi \cdot 2^2 = 4\pi$

$= 2(4\pi) = \boxed{8\pi}$ ✓

4. (a) Let $C = C_1 + C_2 + C_3$

$C_1:\quad x = t \qquad y = t^2 \qquad 0 \le t \le 1$

$\qquad dx = dt \qquad dy = 2t\,dt$

$C_2:\quad x = 1-t \qquad y = 1 \qquad 0 \le t \le 1$

$\qquad dx = -dt \qquad dy = 0$

$C_3:\quad x = 0 \qquad y = 1-t \qquad 0 \le t \le 1$

$\qquad dx = 0 \qquad dy = -dt$

$$\oint_C x^2y^2\,dx + xy\,dy = \int_{C_1} \cdots + \int_{C_2} \cdots + \int_{C_3} \cdots$$

$$= \int_0^1 \left(t^2t^4 + tt^2 \cdot 2t\right) dt + \int_0^1 -(1-t)^2\,dt + \int_0^1 0\,dt$$

$$= \int_0^1 t^6 + 2t^4\,dt + \int_0^1 \left(-1 + 2t - t^2\right) dt$$

$$= \frac{1}{7}t^7 + \frac{2}{5}t^5 - t + t^2 - \frac{1}{3}t^3 \Big|_0^1 = \frac{15 + 42 - 105 + 105 - 35}{105} = \frac{22}{105}$$

(b) $$\int_0^1 \int_{x^2}^1 \left(y - 2x^2y\right) dy\,dx = \int_0^1 \left[\frac{1}{2}y^2 - x^2y^2\right]_{x^2}^1 dx$$

$$= \int_0^1 \left(\frac{1}{2} - x^2 - \frac{1}{2}x^4 + x^6\right) dx = \left[\frac{1}{2}x - \frac{1}{3}x^3 - \frac{1}{10}x^5 + \frac{1}{7}x^7\right]_0^1$$

$$= \frac{105 - 70 - 21 + 30}{210} = \frac{44}{210} = \boxed{\frac{22}{105}}$$ ✓

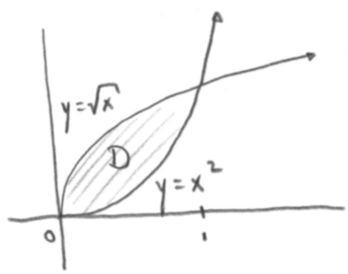
5.

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA \quad , \quad P = xy^2 \ , \ Q = 2x^2y$$

$$= \int_0^2 \int_x^{2x} (4xy - 2xy)\,dy\,dx = 2 \int_0^2 \int_x^{2x} xy\,dy\,dx$$

$$= \cancel{2} \int_0^2 \left[\tfrac{1}{\cancel{2}} xy^2 \right]_x^{2x} dx = \int_0^2 4x^3 - x^3\,dx = \int_0^2 3x^3\,dx = \frac{3}{4}x^4 \Big|_0^2 = \boxed{12}$$

7. $\oint_C (y + e^{\sqrt{x}})\,dx + (2x + \cos y^2)\,dy$



$$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2 - 1)\,dy\,dx = \int_0^1 \sqrt{x} - x^2\,dx$$

$$= \frac{2}{3}x^{3/2} - \frac{1}{3}x^3 \Big|_0^1 = \boxed{\frac{1}{3}}$$

9. $\int_C y^3\,dx - x^3\,dy = \iint_D -3x^2 - 3y^2\,dA = -3 \iint_D x^2 + y^2\,dA$

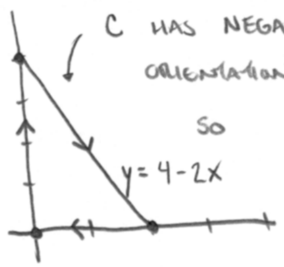
POLAR COORD: $x = r\cos\theta$, $y = r\sin\theta$, $dA = r\,dr\,d\theta$

$$= -3 \int_0^{2\pi} \int_0^2 r^3\,dr\,d\theta = -\frac{3}{4} \int_0^{2\pi} 16\,d\theta = -12(2\pi) = \boxed{-24\pi}$$

10. $$\int_C (1-y^3)\,dx + (x^3+e^{y^2})\,dy = \iint_D \left(3x^2+3y^2\right) dA$$

$$= 3\int_0^{2\pi}\int_2^3 r^3\,dr\,d\theta = \frac{6\pi}{4} r^4 \Big|_2^3 = \frac{3\pi}{2}\left[3^4 - 2^4\right] = \boxed{\frac{195\pi}{2}}$$

11.



$$\oint_C \vec{F}\cdot d\vec{r} = -\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA$$

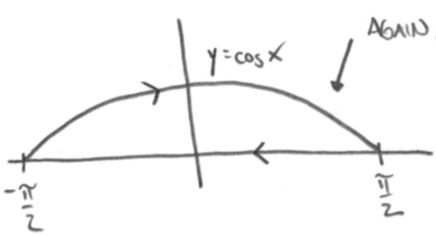
$$= -\int_0^2\int_0^{4-2x} y + \cancel{\cos x} - x\,\cancel{\sin x} - \left(\cancel{\cos x} - x\,\cancel{\sin x}\right) dy\,dx$$

$$= -\int_0^2 \frac{1}{2}y^2\Big]_0^{4-2x} dx = -\int_0^2 \frac{1}{2}(4-2x)^2\,dx = -\int_0^2 8 - 8x + 2x^2\,dx$$

$$= -8x + 4x^2 - \frac{2}{3}x^3 \Big|_0^2 = -16 + 16 - \frac{16}{3} = \boxed{-\frac{16}{3}}$$

12.

AGAIN, C HAS NEGATIVE ORIENTATION:



$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\, dA$$

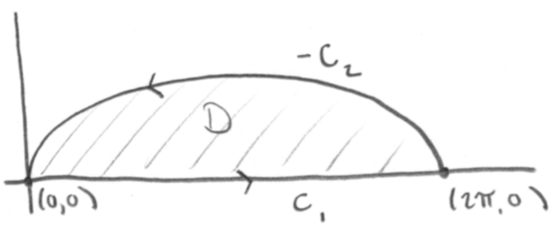
$$= \int_{-\pi/2}^{\pi/2}\int_0^{\cos x} 2y - 2x\,dy\,dx = \int_{-\pi/2}^{\pi/2} y^2 - 2xy\Big|_0^{\cos x} dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 x - 2x\cos x\,dx$$

$$= \int_{-\pi/2}^{\pi/2} \cos^2 x \, dx - \underbrace{\int_{-\pi/2}^{\pi/2} 2x \cos x \, dx}_{\int_{-a}^{a} \text{ODD FUNCTION } dx \,=\, 0}$$

$$= \frac{1}{2} \int_{-\pi/2}^{\pi/2} 1 + \cos(2x) = \frac{1}{2}\left[x + \frac{1}{2}\sin(2x)\right]_{-\pi/2}^{\pi/2} = \boxed{\frac{\pi}{2}}$$

19.



C_1: $x = t \quad y = 0 \quad 0 \le t \le 2\pi$

$dx = dt \quad dy = 0$

C_2: $x = t - \sin t \quad y = 1 - \cos t \quad 0 \le t \le 2\pi$

$dx = (1 - \cos t)\,dt \quad dy = \sin t \, dt$

$$\iint_D 1 \, dA = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\left[\underbrace{\int_{C_1} x\,dy - y\,dx}_{0} + \int_{-C_2} x\,dy - y\,dx\right]$$

$$= \frac{1}{2}\left[-\int_0^{2\pi} (t - \sin t)\sin t - (1 - \cos t)^2 \, dt\right]$$

$$= -\frac{1}{2}\int_0^{2\pi} t\sin t - \sin^2 t - 1 + 2\cos t - \cos^2 t \, dt$$

$$= -\frac{1}{2}\int_0^{2\pi} t\sin t + 2\cos t - 2 \, dt = -\frac{1}{2}\left[\int_0^{2\pi} \overset{u}{t}\, \overbrace{\sin t \, dt}^{dv} + 2\int_0^{2\pi} \cos t - 1 \, dt\right]$$

$$= -\frac{1}{2}\left[-t\cos t\Big|_0^{2\pi} + \int_0^{2\pi} \cos t \, dt + 2(\sin t - t)\Big|_0^{2\pi}\right]$$

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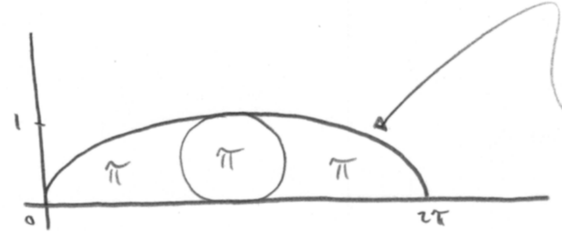
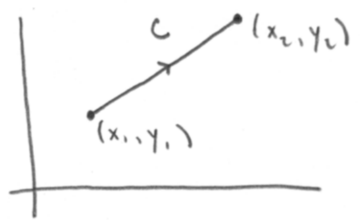
$$= -\frac{1}{2}\left[-2\pi + \underbrace{\sin t\Big|_0^{2\pi}}_{0} + 2(-2\pi)\right] = -\frac{1}{2}\left[-6\pi\right] = \boxed{3\pi}$$



21. (a)



$0 \le t \le 1$

$C: \quad x = (1-t)x_1 + tx_2, \quad y = (1-t)y_1 + ty_2$

$dx = (-x_1 + x_2)\,dt \qquad dy = (-y_1 + y_2)\,dt$

$$\int_C x\,dy - y\,dx = \int_0^1 \Big[(x_1 - tx_1 + tx_2)(y_2 - y_1) - (y_1 - ty_1 + ty_2)(x_2 - x_1)\Big]\,dt$$

$$= \int_0^1 x_1y_2 - \cancel{x_1y_1} - \cancel{tx_1y_2} - \cancel{tx_1y_1} + \cancel{tx_2y_2} - \cancel{tx_2y_1}$$

$$- x_2y_1 + \cancel{x_1y_1} + \cancel{tx_2y_1} - \cancel{tx_1y_1} - \cancel{tx_2y_2} - \cancel{tx_1y_2} \quad dt$$

$$= x_1y_2 - x_2y_1 \qquad \checkmark$$

(b) $A = \iint_D 1\,dA = \frac{1}{2}\int_C y\,dx - x\,dA$ (WHERE C IS THE PERIMETER OF D COUNTERCLOCKWISE) BY GREEN'S THM.

$$= \frac{1}{2}\int_{C_1\ (x_1,y_1)\text{ to }(x_n,y_n)} y\,dx + x\,dy + \dots + \frac{1}{2}\int_{C_n\ (x_n,y_n)\text{ to }(x_1,y_1)} y\,dx + x\,dy$$

$$= \frac{1}{2}\Big[(x_1y_2 - x_2y_1) + \dots + (x_ny_1 - x_1y_n)\Big] \qquad \checkmark$$

(c) $A = \frac{1}{2}\Big[0\cdot 1 - 0\cdot 2 + 2\cdot 3 - 1\cdot 1 + 1\cdot 2 - 3\cdot 0 + 0\cdot 1 - 2(-1) + (-1)\cdot 0 - 1\cdot 0\Big]$

$= \frac{1}{2}\Big[6 - 1 + 2 + 2\Big] = \frac{1}{2}[9] = \boxed{\frac{9}{2}}$

22. GIVEN DENSITY FUNCTION $\rho(x,y)$, $m = \iint_D \rho(x,y)\, dA$

AND $\bar{x} = \frac{1}{m}\iint_D x\,\rho(x,y)\, dA$, $\bar{y} = \frac{1}{m}\iint_D y\,\rho(x,y)\, dA$

HERE WE HAVE $\rho(x,y) = 1$ AND $m = A$

WE HAVE $\bar{x} = \frac{1}{A}\iint_D x\, dA = \frac{1}{A}\iint_D \frac{\partial}{\partial x}\left[\frac{1}{2}x^2\right] - \frac{\partial}{\partial y}[0]\, dA$

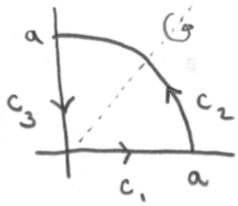
$= \frac{1}{A}\oint_C 0\, dx + \frac{1}{2}x^2\, dy$ BY GREEN'S THM

$= \frac{1}{2A}\oint_C x^2\, dy$ ✓

SIMILARLY, $\bar{y} = \frac{1}{A}\iint_D y\, dA = \frac{1}{A}\iint_D \frac{\partial}{\partial x}[0] - \frac{\partial}{\partial y}\left[-\frac{1}{2}y^2\right] dA$

$= \frac{1}{A}\oint_C -\frac{1}{2}y^2\, dx + 0\, dy$

$= -\frac{1}{2A}\oint_C y^2\, dx$ ✓

23. 

NOTE THAT BY SYMMETRY, $\bar{x} = \bar{y}$.

SO WE HAVE $\bar{y} = \bar{x} = \frac{1}{2A} \oint_C x^2\,dy = \frac{1}{2\left(\frac{\pi a^2}{4}\right)} \left[\int_{C_1} \cdots + \int_{C_2} \cdots + \int_{C_3} \cdots \right]$

C_1: $x = t$ $\quad y = 0 \quad 0 \le t \le a$

$dx = dt \quad dy = 0$

C_2: $x = a\cos t \quad y = a\sin t \quad 0 \le t \le \frac{\pi}{2}$

$dx = -a\sin t\,dt \quad dy = a\cos t\,dt$

C_3: $x = 0 \quad y = a - t \quad 0 \le t \le a$

$dx = 0 \quad dy = -dt$

$$\bar{y} = \bar{x} = \frac{2}{\pi a^2}\left[\underbrace{\int_0^a t^2 \cdot 0}_{0} + \int_0^{\pi/2} a^3\cos^3 t\,dt - \underbrace{\int_0^a 0^2\,dt}_{0} \right.$$

$$= \frac{2a}{\pi}\int_0^{\pi/2} (1 - \sin^2 t)\cos t\,dt \rightsquigarrow \frac{2a}{\pi}\int_0^1 1 - u^2\,du$$

$$= \frac{2a}{\pi}\left[u - \frac{1}{3}u^3\right]_0^1 = \frac{2a}{\pi}\left(\frac{2}{3}\right) = \boxed{\frac{4a}{3\pi}} \quad \text{OR} \quad \boxed{\left(\frac{4a}{3\pi}, \frac{4a}{3\pi}\right)}$$

28. $\oint_C \langle x^2+y, 3x-y^2\rangle \cdot \langle dx, dy\rangle$

$= \oint_C \underbrace{(x^2+y)}_{P}\,dx + \underbrace{(3x-y^2)}_{Q}\,dy$

$= \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\,dA = (3-1)\iint_D dA = 2A = 2(6) = \boxed{12}$