§13.5 Curl & Divergence

Curl: Given $\vec{F} = \langle P, Q, R \rangle$ differentiable vector field on $\mathbb{R}^3$,

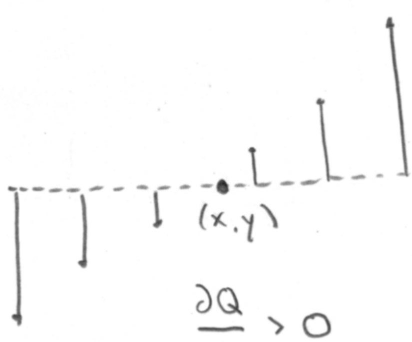
the curl of $\vec{F}$, curl $\vec{F}$, is defined by

$$\text{curl}\,\vec{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

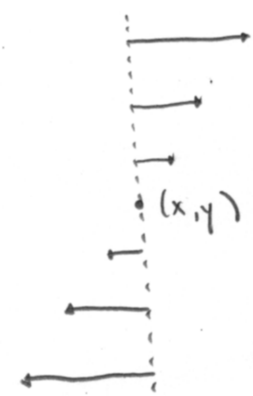
For memory: let $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

then $\boxed{\text{curl}\,\vec{F} = \nabla \times \vec{F}}$

in $\mathbb{R}^2$:



$\frac{\partial Q}{\partial x} > 0$



$\frac{\partial P}{\partial y} > 0$

The difference is a measure of the rotation of the vector field at (x,y)

curl $\vec{F}(x,y,z)$ gives the oriented axis of rotation, and magnitude corresponds to speed of rotation.

PROPERTIES:

- Let $D \subset \mathbb{R}^3$. If $f: D \to \mathbb{R}$ has continuous 2^{nd} partial deriv's,

  then $\quad \text{curl}(\nabla f) = 0 \quad$ (This can be used to show that a vector field is not conservative)

  (Proof:) $\qquad \vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$

- If $\vec{F}: \mathbb{R}^3 \to \mathbb{R}^3$ and component functions have continuous partial deriv's and $\text{curl}\,\vec{F} = \vec{0}$ (everywhere),

  then $\vec{F}$ is conservative. $\qquad F(x,y,z) = \langle y^2 z^3, 2xyz^3, 3xy^2z^2 \rangle$

  (Proof in 13.8)

DIVERGENCE

Given differentiable $\vec{F} = \langle P, Q, R \rangle$ vector field on $\mathbb{R}^3$, the divergence of $\vec{F}$, $\text{div}\,\vec{F}$, is

$$\text{div}\,\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Note: $\text{div}\,\vec{F}$ is a scalar function!

To remember: $\boxed{\text{div}\,\vec{F} = \nabla \cdot \vec{F}}$

THM. If $\vec{F}$ is vector field on $\mathbb{R}^3$ & $\vec{F}$ has cont. 2^{nd} partial derivatives, then

$$\text{div}(\text{curl}\,\vec{F}) = 0.$$

(Proof)

ex. SHOW THAT $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$

CAN'T BE WRITTEN AS CURL OF ANOTHER VECTOR FIELD.

INTERPRETATION: IF $\vec{F}$ IS A VELOCITY FIELD FOR A FLUID OR GAS, THEN DIV $\vec{F}$ REPRESENTS THE NET ~~CHANGE~~ RATE OF CHANGE (WRT TIME) OF THE MASS OF FLUID OR GAS FLOWING FROM THE POINT (x, y, z) PER UNIT VOLUME.

MEASURES THE TENDENCY OF THE FLUID TO DIVERGE FROM THE POINT (x, y, z)

IF DIV $\vec{F} = 0$ THE $\vec{F}$ IS SAID TO BE INCOMPRESSIBLE.

THIRD OPERATION: $\text{DIV}(\nabla f) = f_{xx} + f_{yy} + f_{zz} = \nabla^2 f$

LAPLACE'S EQ: $\nabla^2 f = 0$ (HARMONIC)

VECTOR FORMS OF GREEN'S THM

GIVEN $\vec{F}(x,y)$ ON $\mathbb{R}^2$, REWRITE AS $\vec{F}(x,y,z) = \langle P, Q, 0 \rangle$

THEN $\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = \left\langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$

$s = \int |\vec{r}\,'(t)|\, dt$

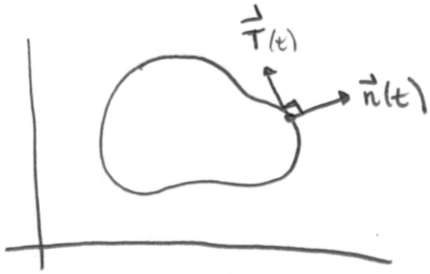
RECALL: $ds = |\vec{r}\,'(t)|\, dt$

SO GREEN'S THM: $\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \hat{k}\, dA$

$\oint_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \vec{T}\, ds$

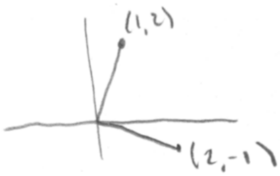
TANGENTIAL COMPONENT.

NORMAL COMPONENT OF $\vec{F}$ ALONG C: $\vec{r}(t) = \langle x(t), y(t) \rangle$, $a \le t \le b$



$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{x'(t)}{|\vec{r}'(t)|} + \frac{y'(t)}{|\vec{r}'(t)|}$$

$$\vec{n}(t) = 90^\circ \text{ CLOCKWISE} \left(\qquad \downarrow \qquad \right)$$



$$= \frac{y'(t)}{|\vec{r}'(t)|} - \frac{x'(t)}{|\vec{r}'(t)|}$$

THEN $$\oint_C \vec{F} \cdot \vec{n} \, ds = \int_a^b (\vec{F} \cdot \vec{n})(t) \, |\vec{r}'(t)| \, dt$$

$$= \int_a^b \left[\frac{P(x(t), y(t)) \, y'(t)}{|\vec{r}'(t)|} , \frac{-Q(x(t), y(t)) \, x'(t)}{|\vec{r}'(t)|} \right] |\vec{r}'(t)| \, dt$$

$$= \int_a^b P \, dy - Q \, dx = \iint_D \underbrace{\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right)}_{\text{DIV } \vec{F}} dA$$

$$\boxed{\oint_C \vec{F} \cdot \vec{n} \, ds = \iint_D \text{DIV } \vec{F} \, dA}$$