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1. curl $\vec{F} = \nabla \times \langle x+yz,\ y+xz,\ z+xy \rangle$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+yz & y+xz & z+xy \end{vmatrix} = \langle x-x,\ y-y,\ z-z \rangle = \boxed{\vec{0}}$$

$$\text{DIV } \vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle x+yz,\ y+xz,\ z+xy \rangle$$

$$= 1 + 1 + 1 = \boxed{3}$$

3. (a) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xye^{z} & 0 & yze^{x} \end{vmatrix} = \boxed{\langle ze^{x},\ xye^{z} - yze^{x},\ -xe^{z} \rangle}$$

(b) $$\nabla \cdot \vec{F} = \boxed{ye^{z} + ye^{x}}$$

7. (a) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{x}\sin y & e^{y}\sin z & e^{z}\sin x \end{vmatrix}$$

$$= \boxed{\langle -e^{y}\cos z,\ -e^{z}\cos x,\ -e^{x}\cos y \rangle}$$

(b) $$\nabla \cdot \vec{F} = \boxed{e^{x}\sin y + e^{y}\sin z + e^{z}\sin x}$$

8. (a) DIV $\vec{F} > 0$

IT APPEARS THAT THE ARROWS POINTING TOWARD A POINT ARE SHORTER THAN THE ARROWS POINTING AWAY FROM A POINT. i.e. MORE FLUID IS FLOWING OUT THAN FLOWING IN.

(ALSO, WE SEE $\frac{\partial P}{\partial x} > 0$ AND $\frac{\partial Q}{\partial y} > 0$!)

(b) CURL $\vec{F} = 0$

LET $\vec{F}(x,y,z) = \langle P(x,y), Q(x,y), 0 \rangle$

SO CURL $\vec{F} = \langle 0, 0, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \rangle$

WE SEE $\frac{\partial Q}{\partial x} = 0$ (ALL ARROWS ALONG $y = c$ HAVE SAME y-COMPONENT)

WE SEE $\frac{\partial P}{\partial y} = 0$ (ALL ARROWS ALONG $x = c$ HAVE SAME x-COMPONENT)

9. (a) DIV $\vec{F} = 0$, $\frac{\partial P}{\partial x} = 0$ AND $\frac{\partial Q}{\partial y} = 0$

(b) CURL $\vec{F} < 0$, $\frac{\partial Q}{\partial x} = 0$ AND $\frac{\partial P}{\partial y} > 0$

10. (a) No. f IS NOT A VECTOR FIELD.

(b) YES. VECTOR FIELD.

(c) YES. SCALAR FIELD

(d) YES. VECTOR FIELD.

(e) NO. WE TAKE GRAD OF SCALARS. $\vec{F}$ IS A VECTOR FIELD.

(f) YES. SCALAR FIELD.

(g) YES. SCALAR FIELD.

(h) NO, WE TAKE DIV OF VECTOR FIELDS ONLY. f IS SCALAR

(i) YES. VECTOR FIELD

(j) NO. DIV $\vec{F}$ IS SCALAR & WE CAN'T TAKE DIV OF SCALARS.

(k) NO. CAN ONLY CROSS VECTORS & DIV $\vec{F}$ IS A SCALAR

(l) YES. SCALAR.

11. SINCE $\text{DOM}(\vec{F}) = \mathbb{R}^3$ AND COMPONENTS HAVE CONTINUOUS PARTIAL DERIVATIVES, $\vec{F}$ IS CONSERVATIVE IF CURL $\vec{F} = \vec{0}$ (CONSTANT).

$$\text{CURL } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle = \vec{0} \ \checkmark$$

$\boxed{\text{YES.}}$ $f_x = y^2z^3 \Rightarrow f(x,y,z) = xy^2z^3 + g(y,z)$

$f_y = 2xyz^3 + g_y(y,z) = 2xyz^3 \Rightarrow g(y,z) = h(z)$

$f_z = 3xy^2z^2 + h'(z) = 3xy^2z^2 \Rightarrow h(z) = C$

$$\boxed{f(x,y,z) = xy^2z^3 + C}$$

13. $$\text{CURL } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy^2z^2 & 2x^2yz^3 & 3x^2y^2z^2 \end{vmatrix} = \langle \underbrace{6x^2yz^2 - 6xy^2z}_{\neq 0}, \dots \rangle$$

$\neq 0$ SO CURL $\vec{F} \neq \vec{0}$.

$\boxed{\text{NOT CONSERVATIVE}}$

17. SINCE THE COMPONENTS OF CURL $\vec{G}$ HAVE CONTINUOUS PARTIAL DERIVATIVES, THE COMPONENTS OF $\vec{G}$ MUST HAVE CONTINUOUS 2nd PARTIAL DERIVATIVES (ASSUMING THERE IS SUCH A $\vec{G}$) AND DIV (CURL $\vec{G}$) MUST EQUAL 0.

IS THIS THE CASE?

$$\text{DIV}(\text{CURL}\,\vec{G}) = \nabla \cdot \langle x \sin y,\ \cos y,\ z - xy \rangle$$
$$= \sin y - \sin y + 1 = 1. \quad \boxed{\text{No!}}$$

19. IRROTATIONAL MEANS CURL IS $\vec{0}$.

$$\text{CURL}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(x) & g(y) & h(z) \end{vmatrix} = \langle 0-0,\ 0-0,\ 0-0 \rangle = \vec{0}$$

20. INCOMPRESSIBLE MEANS DIV IS 0.

$$\text{DIV}\,\vec{F} = \nabla \cdot \langle f(y,z),\ g(x,z),\ h(x,y) \rangle = 0 + 0 + 0 = 0.$$

23. $$\text{DIV}(f\vec{F}) = \nabla \cdot \langle fP,\ fQ,\ fR \rangle = f_x P + f P_x + f_y Q + f Q_y + f_z R + f R_z$$
$$= (f_x P + f_y Q + f_z R) + (f P_x + f P_y + f R_z)$$
$$= \nabla f \cdot \vec{F} + f\ \text{DIV}\,\vec{F}$$

✓

24. $\text{curl}(f\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fP & fQ & fR \end{vmatrix}$

$= \langle f_y R + f R_y - f_z Q - f Q_z, \; f_z P + f P_z - f_x R - f R_x, \; f_x Q + f Q_x - f_y P - f P_y \rangle$

$= f \langle R_y - Q_z, \; P_z - R_x, \; Q_x - P_y \rangle + \langle f_y R - f_z Q, \; f_z P - f_x R, \; f_x Q - f_y P \rangle$

$= f \,\text{curl}\, \vec{F} + (\nabla f) \times \vec{F}$

25. $\text{DIV}(F \times G) = \text{DIV} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ S & T & U \end{vmatrix}$

$G = \langle S, T, U \rangle$

$= \nabla \cdot \langle QU - RT, \; RS - PU, \; PT - QS \rangle$

$= Q_x U + Q U_x - R_x T - R T_x + R_y S + R S_y - P_y U - P U_y + P_z T + P T_z - Q_z S - Q S_z$

$= S(R_y - Q_z) + T(P_z - R_x) + U(Q_x - P_y)$

$- P(U_y - T_z) - Q(S_z - U_x) - R(T_x - S_y)$

$= G \cdot \text{curl}\, \vec{F} - \vec{F} \cdot \text{curl}\, \vec{G}$

26. $\text{DIV}(\nabla f \times \nabla g) = \nabla \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{vmatrix} = \nabla \cdot \langle f_y g_z - f_z g_y, \; f_z g_x - f_x g_z,$

$f_x g_y - f_y g_x \rangle$

$$= f_{yx} g_z + f_y g_{zx} - f_{zx} g_y - f_z g_{yx} + f_{zy} g_x + f_z g_{xy} - f_{xy} g_z - f_x g_{zy}$$

$$+ f_{xz} g_y + f_x g_{yz} - f_{yz} g_x - f_y g_{xz}$$

$$= 0.$$

29. (a) $\nabla r = \nabla \sqrt{x^2 + y^2 + z^2}$

$$= \left\langle \frac{2x}{2\sqrt{x^2+y^2+z^2}}, \frac{2y}{2\sqrt{x^2+y^2+z^2}}, \frac{2z}{2\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}} \langle x, y, z \rangle = \frac{\vec{r}}{r}$$

(b) $\nabla \times \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \langle 0-0, 0-0, 0-0 \rangle = \vec{0}$

(c) $\nabla \frac{1}{r} = \left\langle \frac{\partial}{\partial x}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial y}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right], \frac{\partial}{\partial z}\left[\frac{1}{\sqrt{x^2+y^2+z^2}}\right] \right\rangle$

$$= \left\langle \frac{-2x}{2\sqrt{x^2+y^2+z^2}^3}, \frac{-2y}{2\sqrt{x^2+y^2+z^2}}, \frac{-2z}{2\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$= \frac{1}{\sqrt{x^2+y^2+z^2}^3} \langle -x, -y, z \rangle = \frac{-\vec{r}}{r^3}$$

(d) $\nabla \ln r = \left\langle \frac{\partial}{\partial x}\left[\frac{1}{2}\ln(x^2+y^2+z^2)\right], \dots \right\rangle$

$= \frac{1}{2}\left\langle \frac{2x}{x^2+y^2+z^2}, \frac{2y}{x^2+y^2+z^2}, \frac{2z}{x^2+y^2+z^2} \right\rangle$

$= \frac{\vec{r}}{r^2}$

<u>30.</u> $\text{DIV}\,\vec{F} = \frac{\partial}{\partial x}\left[x(x^2+y^2+z^2)^{-p/2}\right] + \frac{\partial}{\partial y}\left[y(x^2+y^2+z^2)^{-p/2}\right]$

$+ \frac{\partial}{\partial z}\left[z(x^2+y^2+z^2)^{-p/2}\right]$

$= (x^2+y^2+z^2)^{-p/2} - \frac{px}{\cancel{2}}(x^2+y^2+z^2)^{(-p-2)/2} \cdot \cancel{2}x + \dots$

$= (x^2+y^2+z^2)^{\frac{-p-2}{2}}\left((x^2+y^2+z^2) - px^2 + \dots\right)$

$= (x^2+y^2+z^2)^{-\frac{p}{2}}(3-p)$

SO $\text{DIV}\,\vec{F} = 0$ IF $\boxed{p = 3}$

31. GREEN'S THM GIVES

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{DIV} \vec{F}\, dA$$

SETTING $\vec{F} = f(\nabla g)$ WE HAVE

$$\operatorname{DIV} \vec{F} = \operatorname{DIV} \langle f g_x , f g_y , f g_z \rangle$$

$$= f_x g_x + f g_{xx} + f_y g_y + f g_{yy} + f_z g_z + f g_{zz}$$

$$= \nabla f \cdot \nabla g + f \nabla^2 g$$

$$\therefore \oint_C f(\nabla g) \cdot \vec{n}\, ds = \iint_D \left(\nabla f \cdot \nabla g + f \nabla^2 g \right) dA$$

$$= \iint_D \nabla f \cdot \nabla g\, dA + \iint_D f \nabla^2 g\, dA$$

i.e. $$\iint_D f \nabla^2 g\, dA = \oint_C f(\nabla g) \cdot \vec{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA$$