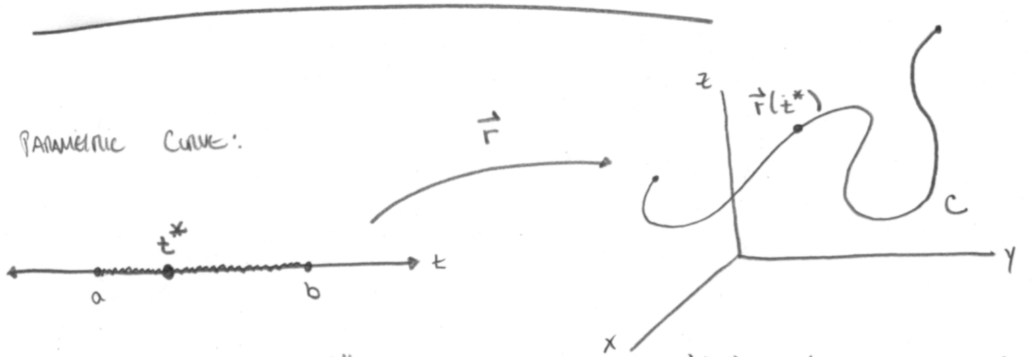
§13.6 Parametric Surfaces & Their Areas

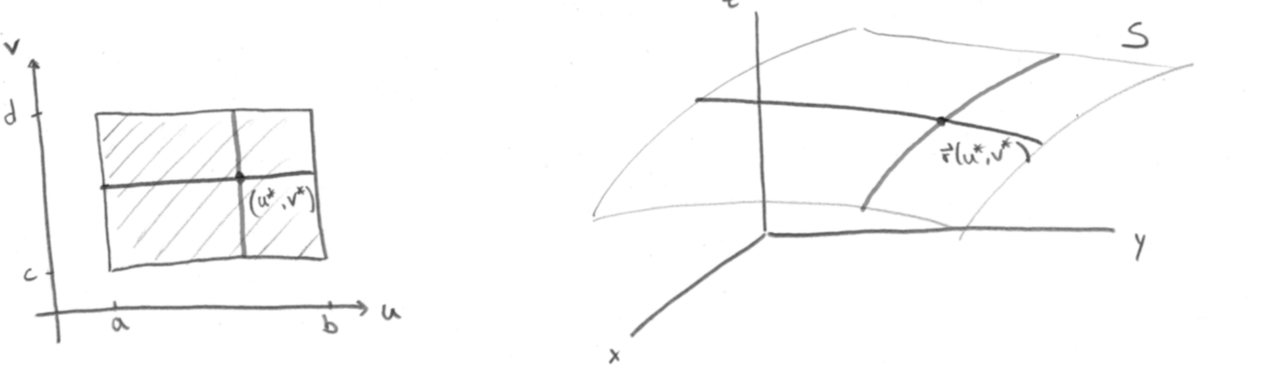
Parametric Curve:



"Perfectly straight"

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

Parametric Surface:



"Perfectly flat"

$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$

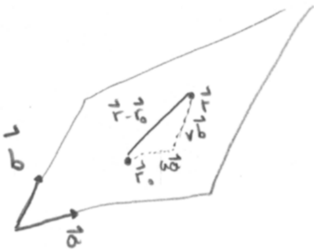
Note that when we fix one variable & let the other vary, $\vec{r}$ becomes a vector function of 1 variable, i.e. curve. Such curves lie on the surface and are called <u>grid curves</u>.

<u>ex.</u> $\vec{r}(t) = \langle 2 \sin u, 3 \cos u, v \rangle$, $0 \le u \le 2\pi$, $0 \le v \le 2$

- identify surface
- sketch
- identify grid lines

- play with domain to make diff. surfaces.

ex. FIND VECTOR FUNCTION THAT PARAMETRIZES THE PLANE THROUGH $\vec{r}_0$ THAT CONTAINS NON-PARALLEL VECTORS $\vec{a}$ & $\vec{b}$.



$$r - r_0 = u\vec{a} + v\vec{b}$$

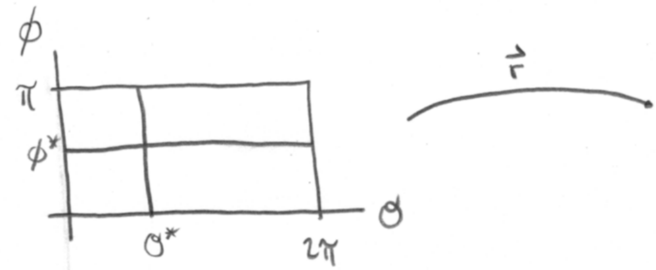
$$\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$$

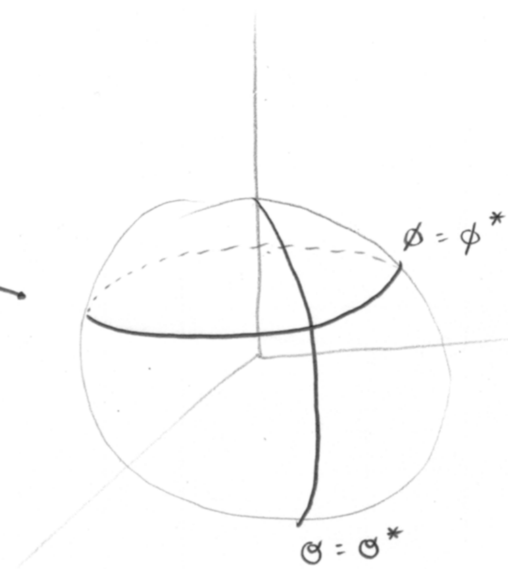
ex: $\vec{r}(u,v) = \langle 7 - 4u + 5v,\ 2 - u - 4v,\ 3 + u + 2v \rangle$ (DOMAIN?)

WHAT IS IT?

ex. SPHERE: $x^2 + y^2 + z^2 = a^2$

SPHERICAL COORD (§ 12.7):



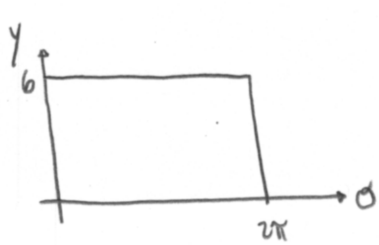


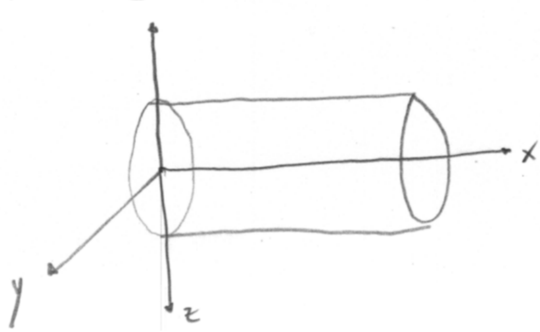
NOTE THAT θ, ϕ ACT AS LONGITUDE, LATITUDE ON SPHERE, RESP.

$$\vec{r}(\theta, \phi) = \langle a \sin\phi \cos\theta,\ a \sin\phi \sin\theta,\ a\cos\phi \rangle$$

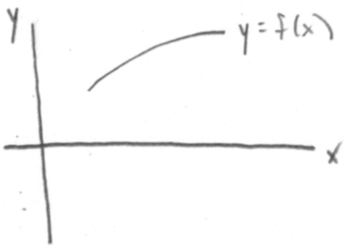
ex. PARAMETRIZE CYLINDER $x^2 + z^2 = 9$, $0 \le y \le 6$

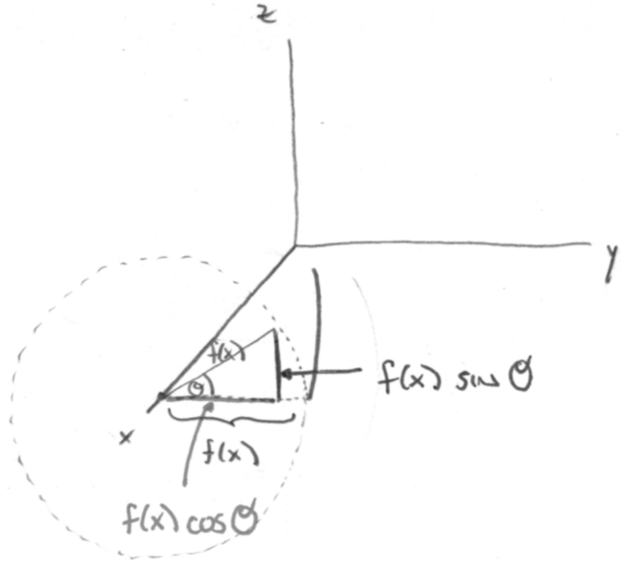
$x = 3\cos\theta$, $y = y$, $z = 3\sin\theta$

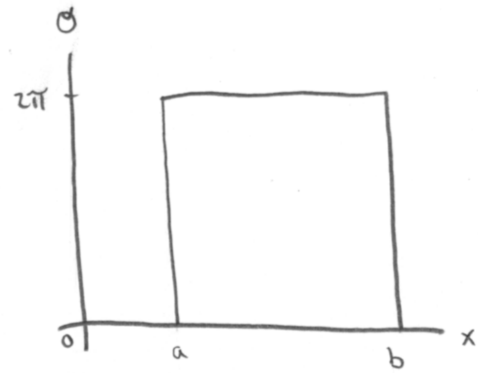


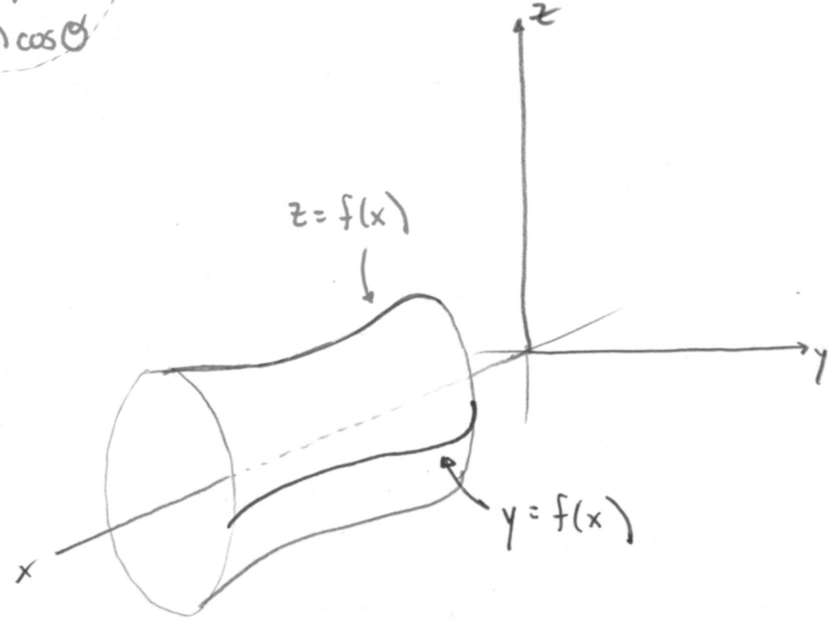


SURFACES OF REVOLUTION







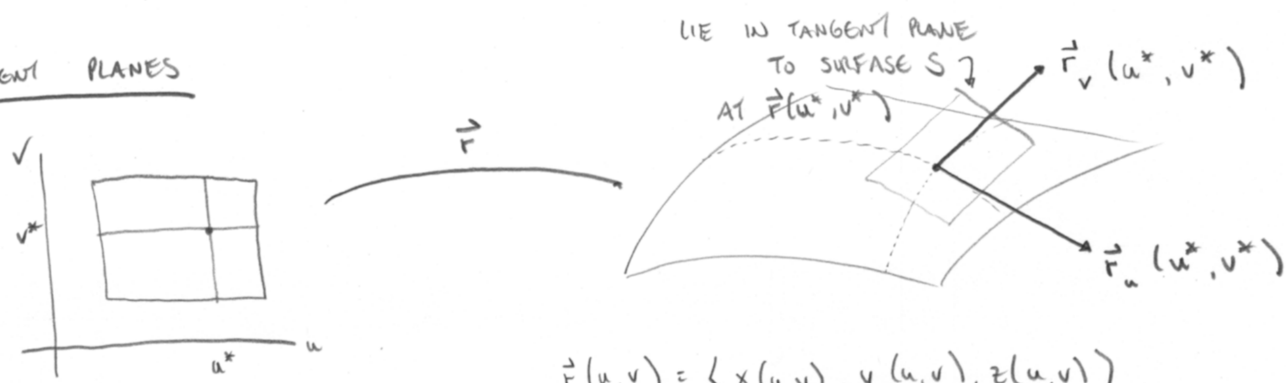


$$\vec{r}(x, \theta) = \langle x, f(x) \cos\theta, f(x) \sin\theta \rangle$$

ex. FIND PARAM. EQ. FOR SURFACE OBTAINED BY ROTATION

$$x = 4y^2 - y^4, \quad -2 \leq y \leq 2, \quad \text{ABOUT } y\text{-AXIS.}$$

TANGENT PLANES



$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

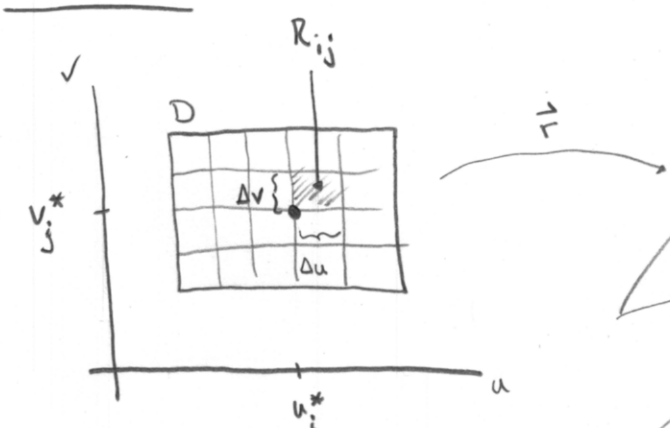
SO LONG AS $\vec{r}_u \times \vec{r}_x \neq \vec{0}$ (IF THIS NEVER HAPPENS, S IS <u>SMOOTH</u>)

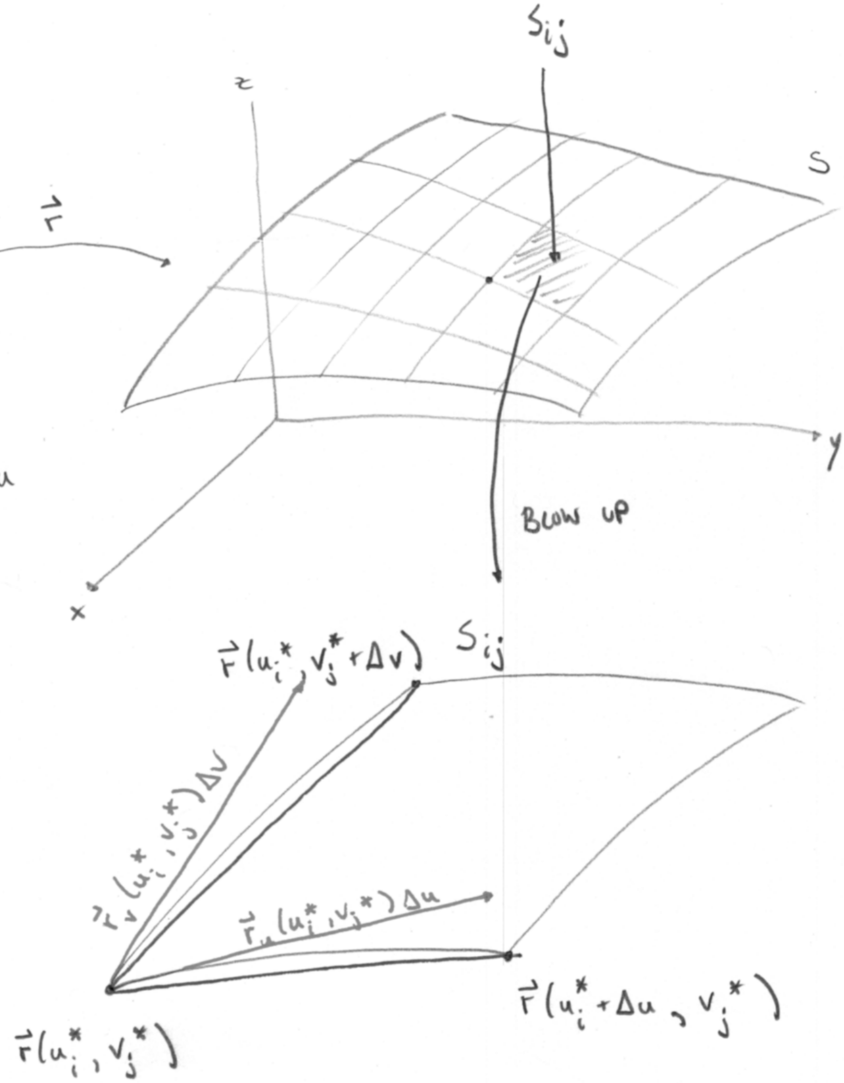
THIS GIVES NORMAL VECTOR TO TANGENT PLANE.

<u>ex.</u> FIND EQ OF TANGENT PLANE TO $\vec{r}(u,v) = \langle u^2+1, v^3+1, u+v \rangle$

AT THE POINT $(5, 2, 3)$.

<u>SURFACE AREA</u>





i COLUMNS

j ROWS

(u_i^*, v_j^*) IS BOTTOM LEFT CORNER OF RECTANGLE R_{ij}

$$\text{AREA}(S_{ij}) \approx \left| \left(\vec{r}(u_i^* + \Delta u, v_j^*) - \vec{r}(u_i^*, v_j^*)\right) \times \left(\vec{r}(u_i^*, v_j^* + \Delta v) - \vec{r}(u_i^*, v_j^*)\right) \right|$$

$$\approx \left| \vec{r}_u(u_i^*, v_j^*)\Delta u \times \vec{r}_v(u_i^*, v_j^*)\Delta v \right|$$

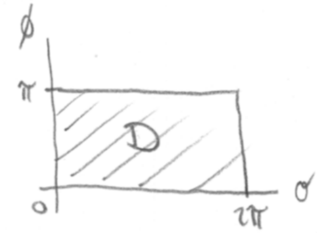
So AREA $S \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \left| \vec{r}_u^{\,*} \times r_v^{\,*} \right| \Delta u \, \Delta j$

$$\boxed{\text{AREA } S = \iint_D \left| \vec{r}_u \times \vec{r}_v \right| dA}$$

NOTE: $dA = du\,dv$ WHEN USING RECT. COORD. SYSTEM TO INTEGRATE OVER D.

ex. FIND SURFACE AREA OF SPHERE $\vec{r}(\theta, \phi) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$

$$A = \iint_D \left| \vec{r}_\phi \times \vec{r}_\theta \right| dA$$



SURFACE AREA OF GRAPH OF FUNCTION

$$\vec{r}(\underbrace{x, y}_{\text{THESE ARE PARAMETERS}}) = \langle x, y, f(x,y) \rangle$$

$$\text{Area}(S) = \iint_D \left| \vec{r}_x \times \vec{r}_y \right| dA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

ex. FIND AREA OF PARABOLOID $z = x^2 + y^2$ THAT LIES UNDER PLANE $z = 9$.

$$A = \int_0^{2\pi}\!\!\int_0^{3} \sqrt{1 + 4r^2}\; r\,dr\,d\theta = \ldots = \frac{\pi}{6}\left(37\sqrt{37} - 1\right)$$