§ 13.6 PARAMETRIC SURFACES & THEIR AREAS

1. $\vec{r}(u,v) = \langle u-v, 3-v, 1+4u+5v \rangle$

$$= \langle 0,3,1 \rangle + u\langle 1,0,4 \rangle + v\langle -1,-1,5 \rangle$$

THIS A PLANE CONTAINING POINT $(0,3,1)$ AND VECTORS $\langle 1,0,4 \rangle$ AND $\langle -1,-1,5 \rangle$.

3. $\vec{r}(s,t) = \langle s, t, t^2 - s^2 \rangle$

THIS IS THE GRAPH $z = f(x,y) = y^2 - x^2$ (HYPERBOLIC PARABOLOID)

11. IV FIXING u GIVES GRID LINES IN SHAPE OF HELIX

FIXING v GIVES GRID LINES THROUGH Z-AXIS PARALLEL TO xy-PLANE.

12. I FIXING u GIVES GRID LINES OF CIRCLES PARALLEL TO xy-PLANE WITH CENTERS ON Z-AXIS.

FIXING v GIVES SINE CURVES THAT PASS THROUGH THE ORIGIN & HEAD OUT RADIALLY IN EVERY DIRECTION

13. II FIXING v GIVES GRID LINES OF CIRCLES PARALLEL TO yz-PLANE WITH CENTERS ON X-AXIS

THE OTHER GRID LINES ARE MORE DIFFICULT TO DESCRIBE

14. III FIXING v GIVES GRID LINES THAT ARE LIKE HELIXES AROUND Z-AXIS, BUT DISTANCE TO Z-AXIS DECREASES AS $u \to 1$ (BOTH $x, y \to 0$ AS AND $z \to 3$ AS $u \to 1$)

FIXING u GIVES GRID LINES THAT ARE CIRCLES THAT LIE IN PLANE $\theta = u$ (CYLINDRICAL/SPERICAL COORD)

16. $\vec{r}(u,v) = \vec{r}_0 + u\vec{a} + v\vec{b}$

$\vec{r}(u,v) = \langle 0,-1,5 \rangle + u\langle 2,1,4 \rangle + v\langle -3,2,5 \rangle$

$$\boxed{\vec{r}(u,v) = \langle 2u-3v,\ -1+u+2v,\ 5+4u+5v \rangle}$$

17. $4x^2 - 4y^2 - z^2 = 4$

$x^2 = 1 + y^2 + \frac{1}{4}z^2$

$x = \sqrt{1 + y^2 + \frac{1}{4}z^2}$ (POSITIVE SQ. RT. SINCE "IN FRONT OF yz-PLANE" $\Rightarrow x \geq 0$)

or

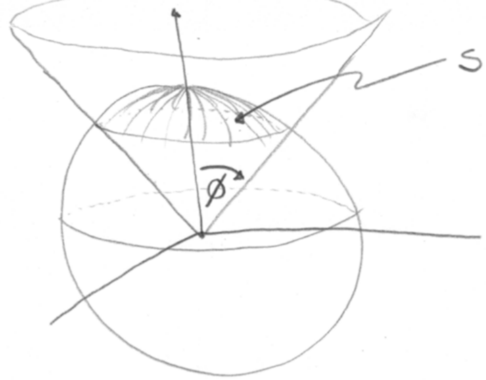
$$\boxed{\vec{r}(y,z) = \left\langle \sqrt{1 + y^2 + \frac{1}{4}z^2},\ y,\ z \right\rangle}$$

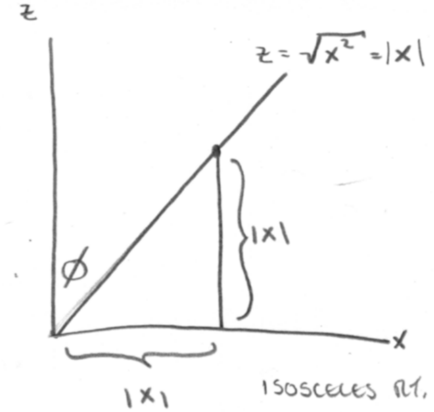
18. LEFT OF xz-PLANE $\Rightarrow y < 0$

$2y^2 = 1 - x^2 - 3z^2$

$y = -\sqrt{\frac{1}{2} - \frac{1}{2}x^2 - \frac{3}{2}z^2}$ or $\boxed{\vec{r}(x,z) = \left\langle x,\ -\sqrt{\frac{1}{2} - \frac{1}{2}x^2 - \frac{3}{2}z^2},\ z \right\rangle}$

19.





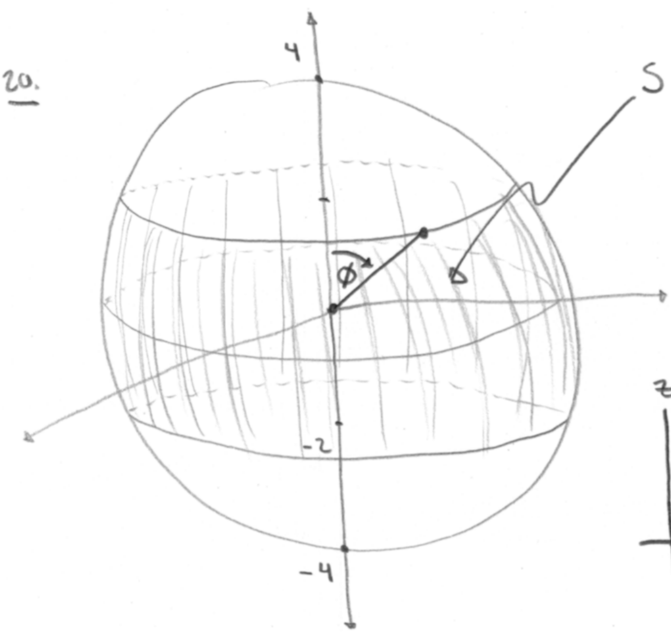
XZ-PLANE $(y=0)$

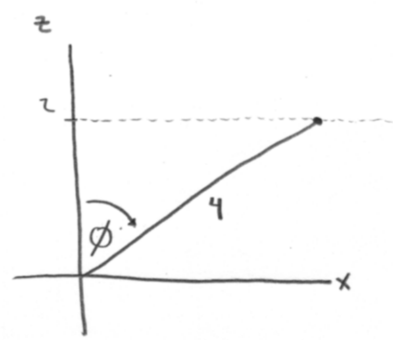
ISOSCELES RT. TRIANGLE

$$\therefore \quad \phi = \frac{\pi}{4}$$

$$\boxed{\begin{gathered} \vec{r}(\theta, \phi) = \langle 2 \sin\phi \cos\theta,\ 2 \sin\phi \sin\theta,\ 2\cos\phi \rangle \\ 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \frac{\pi}{4} \end{gathered}}$$

20.

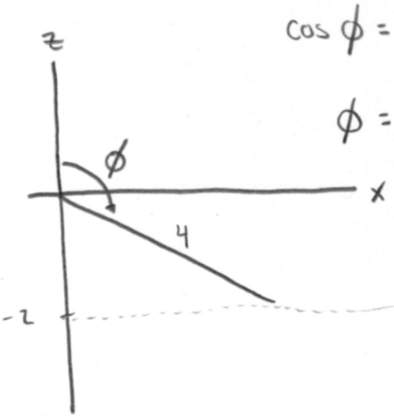




XZ-PLANE $(y=0)$

$$\cos\phi = \frac{2}{4}$$

$$\phi = \frac{\pi}{3}$$



$$\cos\phi = \frac{-2}{4}$$

$$\phi = \frac{2\pi}{3}$$

$$\boxed{\begin{gathered} \vec{r}(\theta, \phi) = \langle 4 \sin\phi \cos\theta,\ 4 \sin\phi \sin\theta,\ 4\cos\phi \rangle \\ 0 \le \theta \le 2\pi, \quad -\frac{\pi}{3} \le \phi \le \frac{\pi}{3} \end{gathered}}$$

22. CYLINDRICAL COORD. $x = r\cos\theta$, $y = r\sin\theta$

$$\boxed{\begin{gathered}\vec{r}(r,\theta) = \langle r\cos\theta\,,\ r\sin\theta\,,\ r\cos\theta + 3\rangle \\ 0 \le r \le 1\ ,\quad 0 \le \theta \le 2\pi\end{gathered}}$$

25. $\vec{r}(x,\theta) = \langle x\,,\ f(x)\cos\theta\,,\ f(x)\sin\theta\rangle$

$$\boxed{\begin{gathered}\vec{r}(x,\theta) = \langle x\,,\ e^{-x}\cos\theta\,,\ e^{-x}\sin\theta\rangle \\ 0 \le x \le 3\ ,\quad 0 \le \theta \le 2\pi\end{gathered}}$$

29. $\vec{r}(u,v) = \langle u+v\,,\ 3u^2\,,\ u-v\rangle$

$\vec{r}_u = \langle 1\,,\ 6u\,,\ 1\rangle$

$\vec{r}_v = \langle 1\,,\ 0\,,\ -1\rangle$

NOTE: $\vec{r}(1,1) = \langle 2,3,0\rangle$

$\therefore$ NORMAL VECTOR TO S AT $(2,3,0)$

$$= \vec{r}_u(1,1) \times \vec{r}_v(1,1) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 6 & 1 \\ 1 & 0 & -1 \end{vmatrix} = \langle -6, 2, -6\rangle$$

EQ OF TANGENT PLANE: $\left(\langle x,y,z\rangle - \langle 2,3,0\rangle\right)\cdot\langle -6,2,-6\rangle = 0$

$-6(x-2) + 2(y-3) - 6z = 0$

$$\boxed{-6x + 2y - 6z = -6}$$

31. $\vec{r}(u,v) = \langle u\cos v, \ u\sin v, \ v \rangle$, $u = 1$, $v = \frac{\pi}{3}$

$\vec{r}_u = \langle \cos v, \ \sin v, \ 0 \rangle$

$\vec{r}_v = \langle -u\sin v, \ u\cos v, \ 1 \rangle$

$\vec{r}(1, \frac{\pi}{3}) = \langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \rangle$

NORMAL VECTOR TO S AT $\vec{r}(1, \frac{\pi}{3})$

$$= \vec{r}_u(1, \tfrac{\pi}{3}) \times \vec{r}_v(1, \tfrac{\pi}{3}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 1 \end{vmatrix}$$

$$= \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 1 \rangle$$

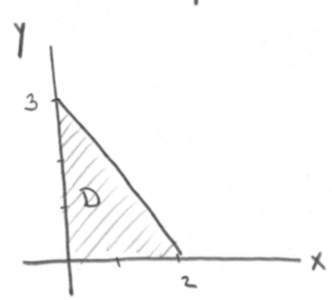
EQ OF TANGENT PLANE: $\left(\langle x, y, z \rangle - \langle \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3} \rangle \right) \cdot \langle \frac{\sqrt{3}}{2}, -\frac{1}{2}, 1 \rangle = 0$

$$\frac{\sqrt{3}}{2}\left(x - \frac{1}{2}\right) - \frac{1}{2}\left(y - \frac{\sqrt{3}}{2}\right) + z - \frac{\pi}{3} = 0$$

$$\frac{\sqrt{3}}{2}x - \frac{1}{2}y + z = \frac{\pi}{3} \quad \text{OR} \quad \boxed{\sqrt{3}x - y + 2z = \frac{2\pi}{3}}$$

33. PLANE INTERSECTS xy-PLANE WHEN $z = 0$

$3x + 2y = 6 \longrightarrow 2y = -3x + 6 \longrightarrow y = -\frac{3}{2}x + 3$



$S: \vec{r}(x,y) = \langle x, y, 6 - 3x - 2y \rangle$

$0 \le x \le 2$, $0 \le y \le -\frac{3}{2}x + 3$

$$|\vec{r}_x \times \vec{r}_y| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -3 \\ 0 & 1 & -2 \end{matrix} \right\| = |\langle 3, 2, 1 \rangle|$$

$$= \sqrt{3^2 + 2^2 + 1^2} = \sqrt{14}$$

$$\therefore \iint_S dS = \iint_D |\vec{r}_x \times \vec{r}_y| \, dA = \int_0^2 \int_0^{-\frac{3}{2}x + 3} \sqrt{14} \, dy \, dx$$

$$= \sqrt{14} \int_0^2 -\frac{3}{2}x + 3 \, dx = \sqrt{14} \left(-\frac{3}{4}x^2 + 3x \right) \Big|_0^2 = \boxed{3\sqrt{14}}$$

35. CYLINDRICAL COORD.: $x = r\cos\theta$, $y = r\sin\theta$

$$x + 2y + 3z = 1 \longrightarrow z = \frac{1}{3}(1 - x - 2y)$$

$$S: \begin{cases} \vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, \frac{1}{3}(1 - r\cos\theta - 2r\sin\theta) \rangle \\ 0 \le r \le \sqrt{3}, \quad 0 \le \theta \le 2\pi \end{cases}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & \frac{1}{3}(-\cos\theta - 2\sin\theta) \\ -r\sin\theta & r\cos\theta & \frac{1}{3}(r\sin\theta - 2r\cos\theta) \end{matrix} \right\|$$

$$= \Big| \Big\langle \frac{1}{3}(r\sin^2\theta - 2r\sin\theta\cos\theta) + \frac{1}{3}(r\cos^2\theta + 2r\sin\theta\cos\theta),$$

$$\frac{1}{3}(r\sin\theta\cos\theta + 2r\sin^2\theta) + \frac{1}{3}(2r\cos^2\theta - r\sin\theta\cos\theta),$$

$$r\cos^2\theta + r\sin^2\theta \Big\rangle \Big|$$

$$= \left| \left\langle \tfrac{1}{3}r, \tfrac{2}{3}r, r \right\rangle \right| = \sqrt{\frac{r^2}{9} + \frac{4r^2}{9} + r^2} = \frac{\sqrt{14}}{3} r$$

$$\text{AREA}(S) = \iint_S dS = \iint_D |\vec{r}_r \times \vec{r}_\theta|\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{\sqrt{14}}{3} r \, dr\, d\theta = 2\pi \cdot \frac{\sqrt{14}}{6} r^2 \Big|_0^{\sqrt{3}} = \pi\sqrt{14}$$

(SEE LAST PAGE OF SOLUTIONS FOR AN EASIER WAY!)

37. $$\iint_S dS = \iint_D \sqrt{(f_x)^2 + (f_y)^2 + 1}\; dA$$

$$= \int_0^1\int_0^1 \sqrt{(x^{1/2})^2 + (y^{1/2})^2 + 1}\; dy\, dx$$

$$= \int_0^1 \frac{2}{3}(x+y+1)^{3/2} \Big|_{y=0}^{y=1} dx$$

$$= \frac{2}{3}\int_0^1 (x+2)^{3/2} - (x+1)^{3/2}\, dx = \frac{2}{3}\cdot\frac{2}{5}\Big[(x+2)^{5/2} - (x+1)^{5/2}\Big]_0^1$$

$$= \frac{4}{15}\Big[3^{5/2} - 2^{5/2} - (2^{5/2} - 1)\Big] = \frac{4}{15}\Big[3^{5/2} - 2\cdot 2^{5/2} + 1\Big]$$

$$= \boxed{\frac{4}{15}\left(3^{5/2} - 2^{7/2} + 1\right)}$$

39. $\vec{r}(x,y) = \langle x, y, xy \rangle$, D = UNIT DISK

$$|\vec{r}_x \times \vec{r}_y| = \left\|\begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & y \\ 0 & 1 & x \end{matrix}\right\| = |\langle -y, -x, 1 \rangle|$$

$$= \sqrt{y^2 + x^2 + 1}$$

$$\text{AREA} = \iint_S dS = \iint_D \sqrt{x^2+y^2+1}\, dA$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\, r\, dr\, d\theta \qquad \begin{aligned} u &= r^2+1 \\ du &= 2r\,dr \end{aligned}$$

$$= \pi \cdot \frac{2}{3}(r^2+1)^{3/2}\Big|_0^1 = \boxed{\frac{2\pi}{3}\left(2^{3/2} - 1\right)}$$

41. $\vec{r}(x,z) = \langle x, 4x+z^2, z \rangle$

$$|\vec{r}_x \times \vec{r}_z| = \sqrt{(f_x)^2 + (f_z)^2 + 1} = \sqrt{4^2 + (2z)^2 + 1}$$

$$= \sqrt{17 + 4z^2}$$

$$\text{AREA} = \iint_S dS = \int_0^1\int_0^1 \sqrt{17+4z^2}\, dx\, dz = \int_0^1 \sqrt{17+4z^2}\, dz$$

$$= 2\int_0^1 \sqrt{\frac{17}{4} + z^2}\, dz \qquad \text{LET } z = \frac{\sqrt{17}}{2}\tan\theta$$

$$dz = \frac{\sqrt{17}}{2}\sec^2\theta\, d\theta$$

$$= 2\int \sqrt{\frac{17}{4} + \frac{17}{4}\tan^2\theta} \cdot \frac{\sqrt{17}}{2}\sec^2\theta \, d\theta$$

$$= \frac{17}{2}\int \sec^3\theta \, d\theta \quad \leftarrow \text{NOPE. I'M NOT DOING THIS.}$$

YOU CAN USE INTEGRATION BY PARTS.

57. $\vec{r}(x,\theta) = \langle x, f(x)\cos\theta, f(x)\sin\theta \rangle$

$$|\vec{r}_x \times \vec{r}_\theta| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{matrix} \right\|$$

$$= |\langle f(x)f'(x)\cos^2\theta + f(x)f'(x)\sin^2\theta, -f(x)\cos\theta, -f(x)\sin\theta \rangle|$$

$$= |\langle f(x)f'(x), -f(x)\cos\theta, -f(x)\sin\theta \rangle|$$

$$= \sqrt{f(x)^2 f'(x)^2 + f(x)^2\cos^2\theta + f(x)^2\sin^2\theta}$$

$$= \sqrt{f(x)^2(f'(x)^2 + 1)} = |f(x)|\sqrt{f'(x)^2 + 1}$$

WE ASSUME $f(x) \geq 0$

SO ABS. VAL. NOT NECESSARY.

$$\therefore \iint_S dS = \int_0^{2\pi}\int_a^b f(x)\sqrt{f'(x)^2 + 1}\, dx\, d\theta = 2\pi\int_a^b f(x)\sqrt{f'(x)^2 + 1}\, dx$$

35.

$z = \frac{1}{3}(1 - x - 2y)$

$\vec{r}(x,y) = \langle x, y, \frac{1}{3}(1-x-2y) \rangle$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{\frac{1}{9} + \frac{4}{9} + 1}$

$= \frac{\sqrt{14}}{3}$

$\therefore$ AREA $= \iint_S dS = \iint_D \frac{\sqrt{14}}{3}\, dA = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{\sqrt{14}}{3}\, r\, dr\, d\theta$

$= 2\pi \frac{\sqrt{14}}{6} r^2 \Big|_0^{\sqrt{3}} = \pi\sqrt{14}$