§ 13.7 SURFACE INTEGRALS

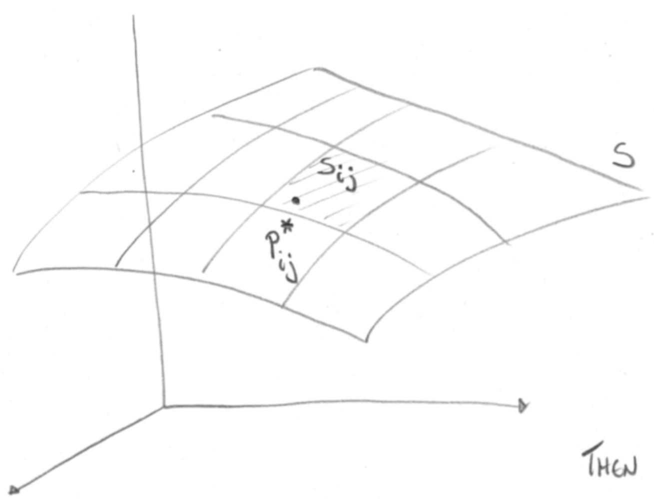
WHAT DOES IT MEAN TO INTEGRATE A FUNCTION OVER A SURFACE?

$$\iint_S f(x,y,z)\,dS$$

IF $f \equiv 1$ THEN THIS IS AREA OF S.

IF $f \geq 0$ REPRESENTS THE DENSITY OF S AT A POINT (x,y,z)

THEN THIS IS THE MASS OF S.



FOR EACH S_{ij}, PICK $P^*_{ij} \in S_{ij}$

AND EVALUATE $f(P^*_{ij})$.

THEN MULTIPLY BY AREA ΔS_{ij}.

THIS GIVES MASS OF S_{ij}.

THEN MASS OF $S \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(P^*_{ij})\,\Delta S_{ij}$

$\lim_{m,n \to \infty}$

AND MASS OF $S = \iint_S f(x,y,z)\,dS$

$$= \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA$$

(AGAIN: WHEN $f \equiv 1$, THIS IS SURFACE AREA.)

COMPARE TO $\int_C f(x,y,z)\,ds = \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$

GRAPHS

e.g. SURFACE IS $z = g(x,y)$

AS WE'VE SEEN, $dS = |\vec{r}_x \times \vec{r}_y| \, dA = \sqrt{g_x^2 + g_y^2 + 1} \, dA$

SO FOR S THE GRAPH $z = g(x,y)$ WE HAVE

$$\boxed{\iint_S f(x,y,z) \, dS = \iint_D f(x,y,g(x,y)) \sqrt{g_x^2 + g_y^2 + 1} \, dA}$$

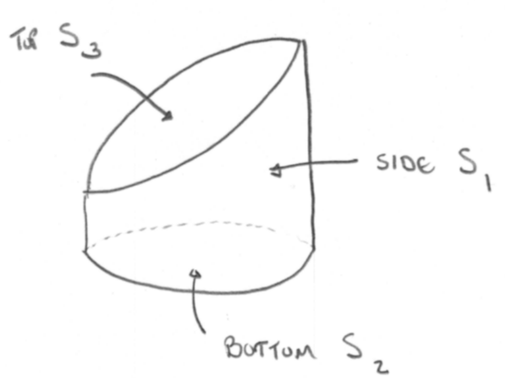
ex. EVALUATE $\iint_S y \, dS$ WHERE S IS SURFACE $z = x + y^2$,

$0 \le x \le 1$, $0 \le y \le 2$

$$\iint_S y \, dS = \iint_D y \sqrt{1 + 4y^2 + 1} \, dA \ \ldots = \frac{13\sqrt{2}}{3}$$

PIECEWISE SMOOTH SURFACE: FINITE UNION OF SMOOTH SURFACES $S_1, S_2, \ldots, S_n$.

$S = S_1 \cup S_2 \cup \ldots \cup S_n$



$$\iint_S f(x,y,z) \, dS = \sum_{i=1}^{n} \iint_{S_i} f(x,y,z) \, dS$$

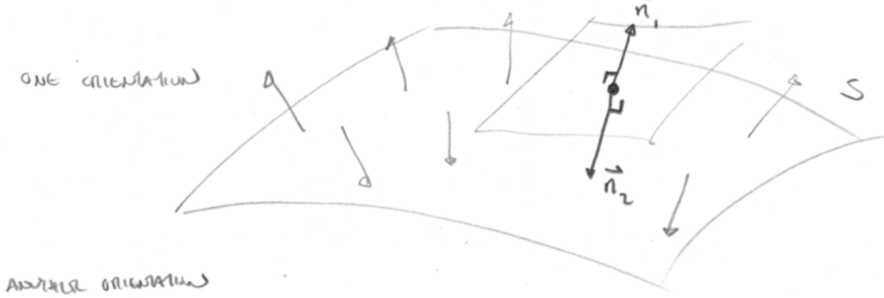
ex. EVALUATE $\iint_S z \, ds$ WHERE $S = S_1 \cup S_2 \cup S_3$

$S_1 : x^2 + y^2 = 1$, $S_2 : z = 0$, $S_3 : z = 1 + x$

ORIENTED SURFACES

2-SIDES SURFACES ARE ORIENTABLE ("TOP" & "BOTTOM")

A SURFACE IS ORIENTED ONCE WE CHOOSE WHICH SIDE IS "TOP".



GIVEN SMOOTH SURFACE S, WE HAVE TANGENT PLANE AT EACH POINT (x, y, z) ON S WITH 2 UNIT NORMAL VECTORS $\vec{n}_1$ & $\vec{n}_2 = -\vec{n}_1$

FOR GRAPH $z = g(x,y)$, WE ORIENT S SO THAT $\vec{n}$ POINTS "UP".

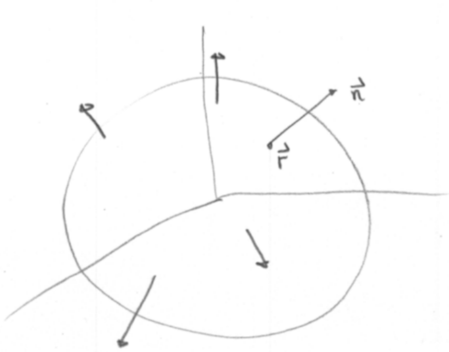
$$\vec{n} = \frac{\vec{r}_x \times \vec{r}_y}{|\vec{r}_x \times \vec{r}_y|} = \frac{\langle -g_x, -g_y, 1 \rangle}{\sqrt{g_x^2 + g_y^2 + 1}}$$

(RECALL: $\hat{i} \times \hat{j} = \underset{\text{UP}}{\hat{k}}$)

(WE COULD ORIENT S IN THE OPPOSITE WAY BY USING $-\vec{n}$) e.g. $z = x^2 + y^2$

MORE GENERALLY, PARAMETRIZED SURFACE S: $\vec{r}(u,v)$

$$\vec{n} = \pm \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$$



e.g. SPHERE $\vec{r}(\phi, \theta) = \langle a \sin\phi \cos\theta, a \sin\phi \sin\theta, a \cos\phi \rangle$

$$\frac{\vec{r}_\phi \times \vec{r}_\theta}{|r_\phi \times r_\theta|} = \frac{\langle a^2 \sin^2\phi \cos\theta, a^2 \sin^2\phi \sin\theta, a^2 \sin^2\phi \cos\phi \rangle}{a^2 \sin\phi}$$

$$= \langle \sin\phi \cos\theta, \sin\phi \sin\theta, \cos\phi \rangle$$

$$= \frac{1}{a} \vec{r}(\phi, \theta)$$

UNIT VECTOR POINTING AWAY FROM ORIGIN "OUTWARD"

NOTE: IF YOU REVERSE THE ORDER OF PARAMETERS, YOU REVERSE THE ORIENTATION OF THE SURFACE:

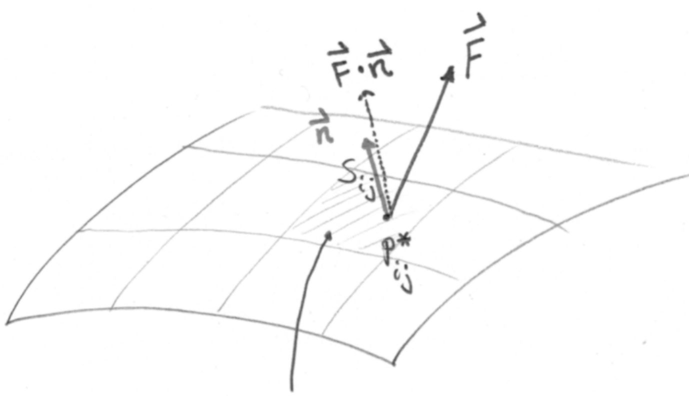
$$\vec{r}_v \times \vec{r}_u = -\vec{r}_u \times \vec{r}_v .$$

FOR CLOSED SURFACE; BY CONVENTION $\vec{n}$ POINTS OUTWARD.

SURFACE INTEGRALS OF VECTOR FIELDS

LET S BE SMOOTH SURFACE IN DOMAIN OF VELOCITY FIELD $\vec{F}$ WHICH DESCRIBES THE FLOW OF A FLUID/GAS.

HOW CAN WE FIND THE VOLUME OF FLUID/GAS THAT PASSES THROUGH S PER UNIT TIME?



THIS CAN BE APPROXIMATED BY

$$\underbrace{\sum_{i=1}^{m}\sum_{j=1}^{n} \vec{F}(P^*_{ij})\cdot\vec{n}(P^*_{ij})}_{\text{COMPONENT OF VELOCITY FIELD } \perp \text{ TO SURFACE}} \Delta S_{ij}$$

$$\xrightarrow[m,n\to\infty]{\text{LIMIT}} \iint_S \vec{F}\cdot\vec{n}\, dS$$

Def: IF $\vec{F}$ IS CONTINUOUS VECTOR FIELD DEFINED ON AN ORIENTED SURFACE S WITH UNIT NORMAL VECTOR $\vec{n}$ THEN THE SURFACE INTEGRAL OF $\vec{F}$ OVER S IS ("FLUX OF $\vec{F}$ ACROSS S")

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\, dS = \iint_D \left(\vec{F}(r(u,v))\cdot\frac{\vec{r}_u\times\vec{r}_v}{|\vec{r}_u\times\vec{r}_v|}\right)|\vec{r}_u\times\vec{r}_v|\, dA$$

$$= \iint_D \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\, dA$$