§15.8 STOKES' THM

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

GREEN'S THM RELATES DOUBLE INTEGRAL OVER PLANE REGION D TO A LINE INTEGRAL ALONG ITS BOUNDARY

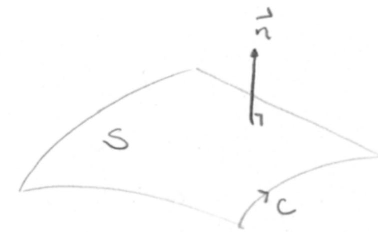
↓ GENERALISED TO PIECEWISE SMOOTH SURFACES

STOKES' THM LET S BE AN ORIENTED PIECEWISE SMOOTH SURFACE BOUNDED BY SIMPLE, CLOSED, PIECEWISE-SMOOTH BOUNDARY CURVE C WITH POSITIVE ORIENTATION*. LET $\vec{F}$ BE VECTOR FIELD WHOSE COMPONENTS HAVE CONTINUOUS PARTIAL DERIV'S ON OPEN REGION IN $\mathbb{R}^3 \supset S$.

THEN

$$\oint_{C} \vec{F} \cdot d\vec{r} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

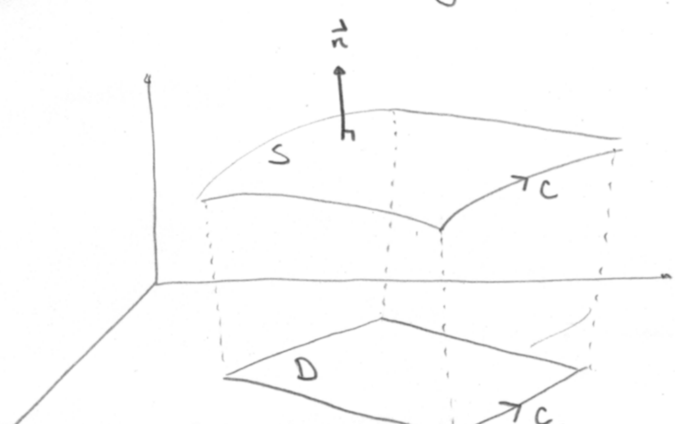
C
$\Updownarrow$
∂S



* C IS POSITIVELY ORIENTED IF WHEN WALKING ALONG C W/ HEAD ON SIDE OF $\vec{n}$, THE SURFACE IS TO YOUR LEFT.

i.e. $$\oint_C \vec{F} \cdot \vec{T}\, ds = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS$$

PROOF (SPECIAL CASE $S: z = g(x,y)$) $\longrightarrow \vec{r}(x,y) = \langle x, y, g(x,y)\rangle$



$$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \iint_D \operatorname{curl}\vec{F}(\vec{r}(x,y))\cdot(\vec{r}_x \times \vec{r}_y)\, dA$$

$$= \iint_D \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle \cdot \langle -g_x, -g_y, 1\rangle\, dA$$

$$= \iint_D \left[\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\left(-\frac{\partial z}{\partial x}\right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\left(-\frac{\partial z}{\partial y}\right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)(1)\right] dA \qquad (*)$$

ON THE OTHER HAND, IF $x = x(t)$, $y = (t)$ PARAMETRIZE C_1 $(a \le t \le b)$

THEN $x = x(t)$, $y = y(t)$, $z = g(x(t), y(t))$ PARAMETRIZE C. $(a \le t \le b)$

WE HAVE (CHAIN RULE:)

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz = \int_a^b \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt$$

$$= \int_a^b \left[P \frac{dx}{dt} + Q \frac{dy}{dt} + R \left(\frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \right) \right] dt$$

$$= \int_a^b \left[\left(P + R \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left(Q + R \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt$$

$$= \int_{C_1} \left(P + R \frac{\partial z}{\partial x} \right) dx + \left(Q + R \frac{\partial z}{\partial y} \right) dy$$

(GREEN'S)

$$= \iint_D \left(\frac{\partial}{\partial x} \left(Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial z}{\partial x} \right) \right) dA$$

$$= \iint_D \left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial x} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial z \partial y} \right.$$
$$\left. - \left(\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + \frac{\partial R}{\partial y} \frac{\partial z}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial z} \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial y \partial x} \right) \right) dA$$

$= (*)$

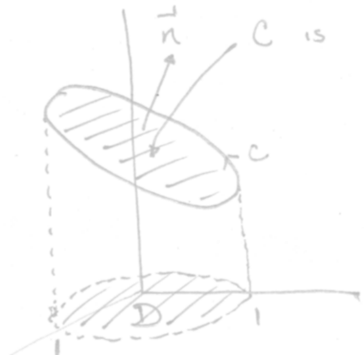
$\square$

ex. EVALUATE $\int_C \vec{F} \cdot d\vec{r}$ WHERE $\vec{F}(x,y,z) = \langle -y^2, x, z^2 \rangle$

AND C IS CURVE OF INTERSECTION OF PLANE $z = 2 - y$ &

CYLINDER $x^2 + y^2 = 1$, ORIENTED C.C. WHEN VIEWED FROM ABOVE.

ANS: C IS BOUNDARY TO THIS SURFACE (NOT UNIQUE)



$$\int_C \vec{F} \cdot d\vec{r} \overset{\text{S.T.}}{=} \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \ldots$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1 - 2y \rangle$$

$S: \ \vec{r}(x,y) = \langle x, y, 2 - y \rangle$

$$d\vec{S} = (\vec{r}_x \times \vec{r}_y)\, dA = \langle 0, 1, 1 \rangle\, dA$$

$$\ldots = \iint_D \langle 0, 0, 1-2y \rangle \cdot \langle 0, 1, 1 \rangle\, dA = \iint_D 1 - 2y \ dA$$

$$= \int_0^{2\pi} \int_0^1 r - 2r^2 \sin\theta \ dr\, d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{2}{3}\sin\theta \ d\theta = \boxed{\pi}$$

$$\frac{1}{2}r^2 - \frac{2}{3}r^3 \sin\theta \Big]_{r=0}^{r=1}$$

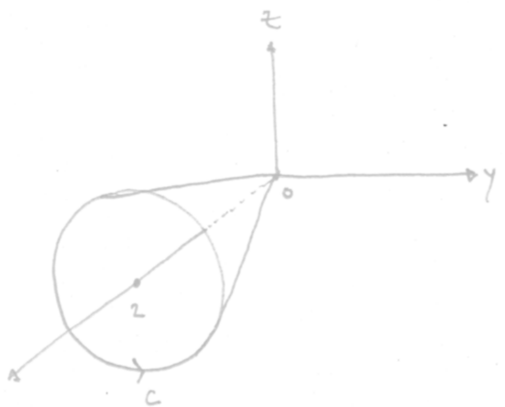
$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

ex. EVALUATE ~~$\iint_S \vec{F} \cdot d\vec{S}$~~ WHEN $\vec{F}(x,y,z) = \langle \tan^{-1}(x^2yz^2),\ x^2y,\ x^2z^2 \rangle$

& S IS CONE $x = \sqrt{y^2+z^2}$, $0 \le x \le 2$ ORIENTED IN

DIRECTION OF POSITIVE X-AXIS.

ANS.



S.T.

$$\ldots = \int_C \vec{F} \cdot d\vec{r} = \ldots$$

$\vec{r}(\theta) = \langle 2,\ 2\cos\theta,\ 2\sin\theta \rangle, \quad 0 \le \theta \le 2\pi$

$\vec{r}\,'(\theta) = \langle 0,\ -2\sin\theta,\ 2\cos\theta \rangle$

$$\ldots = \int_0^{2\pi} \vec{F}(\vec{r}(\theta)) \cdot \vec{r}\,'(\theta)\, d\theta = \int_0^{2\pi} \langle \ldots,\ 8\cos\theta,\ 16\sin^2\theta \rangle \cdot \langle 0,\ -2\sin\theta,\ 2\cos\theta \rangle\, d\theta$$

$$= \int_0^{2\pi} -16\sin\theta\cos\theta + 32\sin^2\theta\cos\theta\ d\theta$$

LET $u = \sin\theta$
$du = \cos\theta\, d\theta$

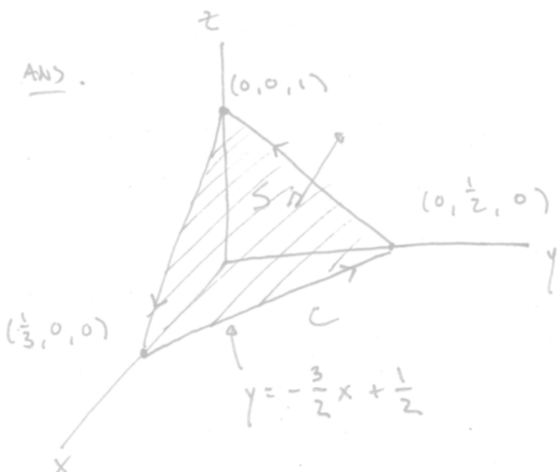
$$= -8\sin^2\theta + \frac{32}{3}\sin^3\theta \Big|_0^{2\pi} = \boxed{0}$$

ex. FIND $\int_C \vec{F}\cdot d\vec{r}$ WHEN $\vec{F}(x,y,z) = \langle 1, x+yz, xy-\sqrt{z}\rangle$

& C IS BOUNDARY OF PART OF PLANE $3x+2y+z=1$

IN 1^{st} OCTANT, CC VIEWED FROM ABOVE

ANS.



$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \dots$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix}$$

$$= \langle x-y, -y, 1\rangle$$

$$S: \vec{r}(x,y) = \langle x, y, 1-3x-2y\rangle \quad ; \quad d\vec{S} = (\vec{r}_x \times \vec{r}_y)\, dA$$

$$\vec{r}_x \times \vec{r}_y = \langle 3, 2, 1\rangle$$

$$\dots = \iint_D \overbrace{\langle x-y, -y, 1\rangle \cdot \langle 3,2,1\rangle}^{3x-3y-2y+1}\, dA$$

$$= \int_0^{1/3}\int_0^{-\frac{3}{2}x+\frac{1}{2}} 3x-5y+1\; dy\,dx = \int_0^{1/3} -\frac{81}{8}x^2 + \frac{15}{4}x - \frac{1}{8}\; dx$$

$$\left(\begin{array}{l} (1+3x)\left(-\frac{3}{2}x+\frac{1}{2}\right) - \frac{5}{2}\left(-\frac{3}{2}x+\frac{1}{2}\right)^2 \\ \cancel{-\frac{3}{2}x} + \frac{1}{2} - \frac{9}{2}x^2 + \cancel{\frac{3}{2}x} \quad -\frac{5}{2}\left(\frac{9}{4}x^2 - \frac{3}{2}x + \frac{1}{4}\right) \\ \qquad -\frac{45}{8}x^2 + \frac{15}{4}x - \frac{5}{8} \end{array} \right)$$

$$= -\frac{27}{8}x^3 + \frac{15}{8}x^2 - \frac{1}{8}x \Big|_0^{1/3}$$

$$= -\frac{1}{8} + \frac{5}{24} - \frac{1}{24} = -\frac{1}{8} + \frac{1}{6}$$

$$= \frac{2}{48} = \boxed{\frac{1}{24}}$$

13.8.5