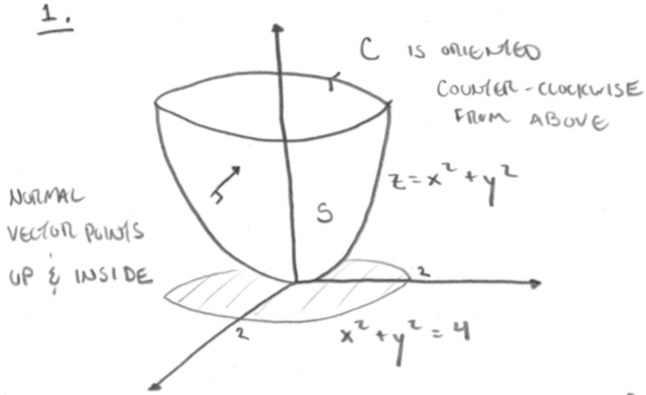
§13.8 STOKES THM

1.



$C: \vec{r}(t) = \langle 2\cos t, 2\sin t, 4 \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$

$0 \le \theta \le 2\pi$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

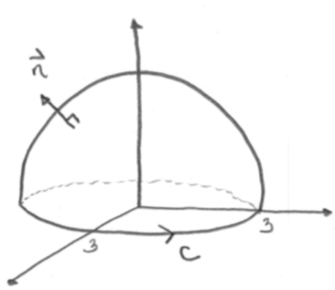
$$= \int_C x^2z^2\,dx + y^2z^2\,dy + xyz\,\underbrace{dz}_{0}$$

$$= \int_0^{2\pi} \Big((2\cos t)^2 (4)^2 (-2\sin t) + (2\sin t)^2 (4)^2 (2\cos t) \Big)\,dt$$

$$= 128 \int_0^{2\pi} \left(-\cos^2 t \sin t + \sin^2 t \cos t \right) dt \qquad (\text{2 DIFFERENT } u\text{-SUBS})$$

$$= 128 \left[\frac{1}{3}\cos^3 t + \frac{1}{3}\sin^3 t \right]_0^{2\pi} = \boxed{0}$$

2.



$C: \vec{r}(t) = \langle 3\cos t, 3\sin t, 0 \rangle, \quad 0 \le t \le 2\pi$

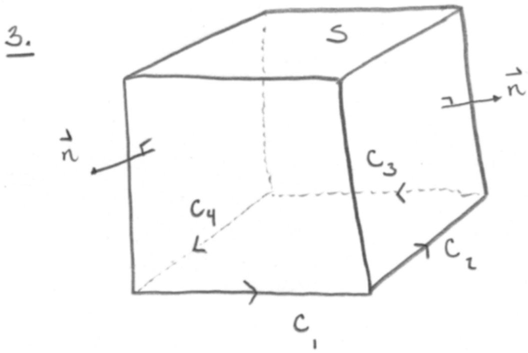
$\vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 0 \rangle$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_C 2y\underbrace{\cos z}_{1}\,dx + e^x \underbrace{\sin z}_{0}\,dy + xe^y\,\underbrace{dz}_{0} = \int_C 2y\,dx = \int_0^{2\pi} 2(3\sin t)(-3\sin t)\,dt$$

$$= \int_0^{2\pi} -18\sin^2 t\,dt = -9\int_0^{2\pi} (1 - \cos(2t))\,dt = \boxed{-18\pi}$$

3.



C COSISTS OF FOUR LINE SEGMENTS, CONNECTING THE POINTS

C_1: $(1,-1,-1)$ TO $(1,1,-1)$,

C_2: $(1,1,-1)$ TO $(-1,1,-1)$,

C_3: $(-1,1,-1)$ TO $(-1,-1,-1)$, AND

C_4: $(-1,-1,-1)$ TO $(1,-1,-1)$

FOR EACH LINE SEGMENT, ONE COORDINATE CHANGES EITHER FROM -1 TO 1 OR FROM 1 TO -1 WHILE THE OTHERS REMAIN FIXED.

e.g. C_1 : $\vec{r}(t) = \langle 1, y, -1 \rangle$, $-1 \le y \le -1$

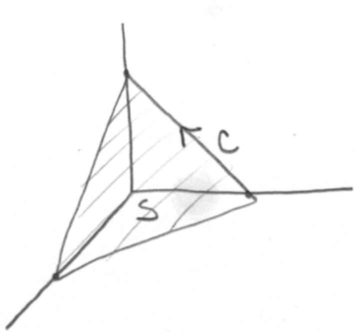
(NOTE THAT $dx = dz = 0$ WHILE $dy = 1$.)

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_{C_1} xy\,dy + \int_{C_2} xyz\,dx + \int_{C_3} xy\,dy + \int_{C_4} xyz\,dx$$

$$= \int_{-1}^{1} y\,dy + \int_{1}^{-1} -x\,dx + \int_{1}^{-1} -y\,dy + \int_{-1}^{1} x\,dx$$

$$= \tfrac{1}{2}y^2\Big|_{-1}^{1} - \tfrac{1}{2}x^2\Big|_{1}^{-1} - \tfrac{1}{2}y^2\Big|_{1}^{-1} + \tfrac{1}{2}x^2\Big|_{-1}^{1} = \boxed{0}$$

5.



S CONTAINS VECTORS $\hat{j} - \hat{i} = \langle -1, 1, 0 \rangle = \vec{a}$

AND $\hat{k} - \hat{i} = \langle -1, 0, 1 \rangle = \vec{b}$

SO NORMAL VECTOR TO PLANE S IS

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

SO $S: \; x + y + z = D$ AND PLUGGING IN ANY KNOWN POINT ON S (e.g. $(1,0,0)$) WE HAVE

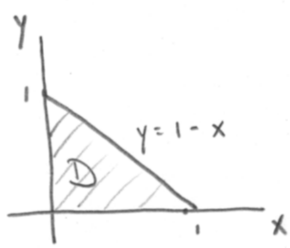
$$S: z = 1 - x - y = g(x,y) \text{ (GRAPH)}.$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \ldots$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y^2 & y+z^2 & z+x^2 \end{vmatrix} = \langle -2z, -2x, -2y \rangle$$

$z = 1 - x - y$

$$d\vec{S} = \langle -g_x, -g_y, 1 \rangle dA = \langle 1, 1, 1 \rangle dA$$



$$\ldots = \int_0^1 \int_0^{1-x} \langle -2 + 2x + 2y, -2x, -2y \rangle \cdot \langle 1, 1, 1 \rangle \, dy\,dx$$

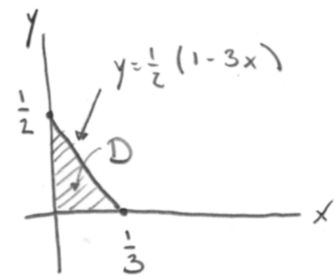
$$= \int_0^1 \int_0^{1-x} -2 \, dy\,dx = -2 \int_0^1 1 - x \, dx = -2\left(x - \tfrac{1}{2}x^2\right)\Big|_0^1 = \boxed{-1}$$

6. PLANE INTERSECTS xy-PLANE WHEN $z = 0$:

$$3x + 2y = 1 \longrightarrow y = \tfrac{1}{2}(1 - 3x)$$

PLANE: $z = 1 - 3x - 2y = g(x,y)$ (GRAPH)

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & x+yz & xy-\sqrt{z} \end{vmatrix} = \langle x-y\,,\,-y\,,\,1 \rangle$$



$$d\vec{S} = \langle -g_x\,,\,-g_y\,,\,1 \rangle\, dA = \langle 3\,,\,2\,,\,1 \rangle\, dA$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \langle x-y\,,\,-y\,,\,1\rangle \cdot \langle 3,2,1\rangle\, dA$$

$$3x-3y-2y+1$$

$$= \int_0^{1/3}\int_0^{\frac{1}{2}(1-3x)} (3x-5y+1)\,dy\,dx = \int_0^{1/3} 3xy - \frac{5}{2}y^2 + y \Big|_{y=0}^{y=\frac{1}{2}(1-3x)} dx$$

$$3x\cdot\frac{1}{2}(1-3x) - \frac{5}{2}\cdot\frac{1}{4}(1-3x)^2 + \frac{1}{2}(1-3x)$$

$$\cancel{\frac{3}{2}x} - \frac{9}{2}x^2 - \frac{5}{8} + \frac{15}{4}x - \frac{45}{8}x^2 + \frac{1}{2} - \cancel{\frac{3}{2}x}$$

$$= \int_0^{1/3} \frac{9}{8}x^2 + \frac{15}{4}x - \frac{1}{8}\,dx$$

$$= \frac{3}{8}x^3 + \frac{15}{8}x^2 - \frac{1}{8}x \Big|_0^{1/3} = \frac{1}{72} + \frac{5}{24} - \frac{1}{24} = \frac{1+15-3}{72} = \boxed{\frac{13}{72}}$$

7. $S: \vec{r}(x,y) = \langle x\,,\,y\,,\,5 \rangle\,,\quad x^2+y^2 \le 16$

$$\vec{r}_x \times \vec{r}_y = \langle 0\,,\,0\,,\,1 \rangle$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & 2xz & e^{xy} \end{vmatrix} = \langle xe^{xy}-2x\,,\,y-ye^{xy}\,,\,z \rangle$$

$$\int_C \vec{F}\cdot dr = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \langle xe^{xy}-2x\,,\, y-ye^{xy}\,,\, \underset{\substack{\uparrow \\ z=5}}{z}\rangle\cdot\langle 0,0,1\rangle\, dA$$

$$= \iint_D z\, dA = 5\iint_D dA = 5(16\pi) = \boxed{80\pi}$$

8. $S: \vec{r}(x,y) = \langle x\,,\, y\,,\, 5-x\rangle \,,\quad x^2+y^2 \le 9$

$$\vec{r}_x \times \vec{r}_y = \langle 1\,,\, 0\,,\, 1\rangle$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2z & 3y \end{vmatrix} = \langle 1\,,\, 0\,,\, -x\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \langle 1,0,-x\rangle\cdot\langle 1,0,1\rangle\, dA$$

$$= \iint_D 1-x\;dA \longrightarrow \int_0^{2\pi}\int_0^{3} (1-r\cos\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{3}r^3\cos\theta \,\Big|_{r=0}^{r=3} d\theta = \int_0^{2\pi} \frac{9}{2} - 9\cos\theta\; d\theta = \boxed{9\pi}$$

10.a. $\vec{r}(x,y) = \langle x, y, y^2 - x^2 \rangle$, $x^2 + y^2 \le 1$

$\vec{r}_x \times \vec{r}_y = \langle 2x, -2y, 1 \rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & \frac{1}{3}x^3 & xy \end{vmatrix} = \langle x, -y, 0 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \langle x, -y, 0 \rangle \cdot \langle 2x, -2y, 1 \rangle \, dA$$

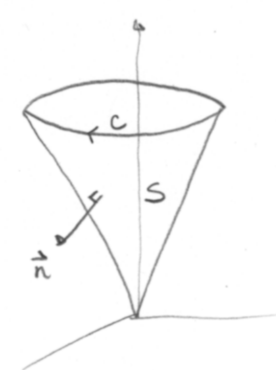
$$= \iint_D 2x^2 + 2y^2 \, dA = 2 \int_0^{2\pi} \int_0^1 r^3 \, dr \, d\theta = \boxed{\pi}$$

11. $C: \vec{r}(t) = \langle 4\cos t, -4\sin t, 4 \rangle$ (CLOCKWISE) , $0 \le t \le 2\pi$

$\vec{r}\,'(t) = \langle -4\sin t, -4\cos t, 0 \rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle 4\sin t, 4\cos t, -2 \rangle \cdot \langle -4\sin t, -4\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} \left(-16\sin^2 t - 16\cos^2 t \right) dt = -16(2\pi) = \boxed{-32\pi}$$



ON THE OTHER HAND,

$S: \vec{r}(r, \theta) = \langle r\cos\theta, r\sin\theta, r \rangle$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle -r\cos\theta, -r\sin\theta, r \rangle$$

SINCE $r > 0$, THIS IS ORIENTED UPWARD. SO WE USE $\vec{r}_\theta \times \vec{r}_r = -(\vec{r}_r \times \vec{r}_\theta)$ INSTEAD.

13.8.6

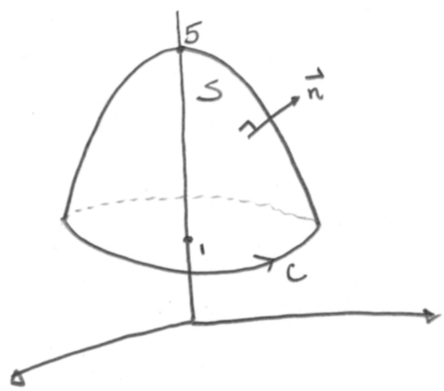
$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & -z \end{vmatrix} = \langle 0, 0, 2 \rangle$$

$$\iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^4 \langle 0, 0, 2\rangle \cdot \langle r\cos\theta, r\sin\theta, -r\rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^4 -2r \, dr\, d\theta = 2\pi \cdot -r^2 \Big|_0^4 = \boxed{-32\pi}$$

✓

12.



$C: \vec{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0\rangle$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle -4\sin t, 2\sin t, 6\cos t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle \, dt$$

$$= \int_0^{2\pi} 8\sin^2 t + 4\sin t\cos t \, dt = 8\pi + 0 = \boxed{8\pi}$$

ON THE OTHER HAND, $S: \vec{r}(x,y) = \langle x, y, 5 - x^2 - y^2\rangle$

$\vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1\rangle$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2yz & y & 3x \end{vmatrix} = \langle 0, -2y-3, 2z\rangle$$

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_D \langle 0, -2y-3, 2z\rangle \cdot \langle 2x, 2y, 1\rangle \, dA$$

$$= \iint_D -4y^2 - 6y + 2z \, dA = \iint_D \left(-6y^2 - 2x^2 + 10 - 6y\right) dA$$

$$z = 5 - x^2 - y^2 \longrightarrow -4y^2 - 6y + 10 - 2x^2 - 2y^2$$

$$= \int_0^{2\pi}\int_0^2 \left(-6r^2\sin^2\theta - 2r^2\cos^2\theta + 10 - 6r\sin\theta\right) r\,dr\,d\theta$$

$$-\frac{6}{4}r^4\sin^2\theta - \frac{2}{4}r^4\cos^2\theta + \frac{10}{2}r^2 - \frac{6}{3}r^3\sin\theta \Big|_{r=0}^{r=2}$$

$$= \int_0^{2\pi} -24\sin^2\theta - 8\cos^2\theta + 20 - 16\sin\theta \, d\theta$$

$$= -24\pi - 8\pi + 40\pi - 0 = \boxed{8\pi}$$

14. $\int_C z\,dx - 2x\,dy + 3y\,dz = \int_C \vec{F}\cdot d\vec{r}$

WHERE $\vec{F}(x,y,z) = \langle z, -2x, 3y\rangle$

THEN $\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & -2x & 3y \end{vmatrix} = \langle 3, 1, -2\rangle$

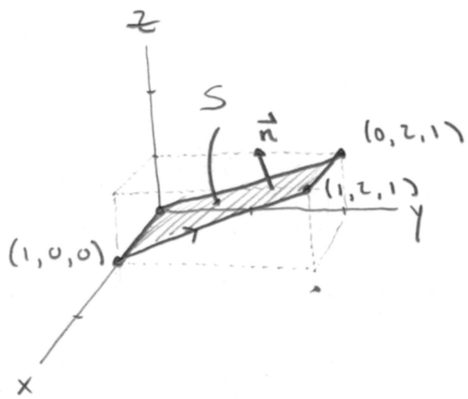
C IS BOUNDARY OF S. SINCE C LIES IN PLANE $x+y+z=1$, WE CAN CHOOSE S TO ALSO LIE IN THE SAME PLANE. THIS PLANE HAS NORMAL VECTOR $\langle 1,1,1\rangle$, AND UNIT NORMAL VECTOR $\vec{n} = \frac{1}{\sqrt{3}}\langle 1,1,1\rangle$.

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$$

$$= \frac{1}{\sqrt{3}}\iint_S \langle 3,1,-2\rangle\cdot\langle 1,1,1\rangle\,dS = \frac{2}{\sqrt{3}}\iint_S dS$$

$= \frac{2}{\sqrt{3}} \times$ SURFACE AREA OF PLANAR REGION ENCLOSED BY C ✓

15.



$$\vec{F} = \langle z^2, 2xy, 4y^2\rangle$$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & 2xy & 4y^2 \end{vmatrix} = \langle 8y, 2z, 2y\rangle$$

$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \ldots$ S CONTAINS VECTORS $\langle 1,0,0\rangle$ AND $\langle 0,2,1\rangle$

SO A NORMAL VECTOR IS $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 0 \\ 0 & 2 & 1 \end{vmatrix} = \langle 0,-1,2\rangle$

$$\downarrow$$

$$-y + 2z = 0$$

$$z = \frac{1}{2}y$$

$S: \vec{r}(x,y) = \langle x, y, \frac{1}{2}y\rangle$

$$\vec{r}_x \times \vec{r}_y = \langle 0, -\frac{1}{2}, 1\rangle$$

$$\ldots = \iint_D \langle 8y, 2z, 2y \rangle \cdot \langle 0, -\tfrac{1}{2}, 1 \rangle \, dA$$

$$2z = y$$

$$= \iint_D \tfrac{3}{2} y \, dA = \int_0^1 \int_0^2 \tfrac{3}{2} y \, dy \, dx = \tfrac{3}{4} y^2 \Big|_0^2 = \boxed{3}$$

17. SINCE A SPHERE S HAS NO BOUNDARY,

$$\int_{\partial S} \vec{F} \cdot d\vec{r} = \int_{\emptyset} \vec{F} \cdot d\vec{r} = 0.$$

BOUNDARY OF SPHERE. EMPTY SET.

$$\therefore \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \int_{\partial S} \vec{F} \cdot d\vec{r} = 0.$$