§ 13.9 THE DIVERGENCE THM

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4/19/2017

RECALL GREEN'S THM (2D) (§13.4)

I $\displaystyle\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\underbrace{\quad C \qquad\qquad D \quad}_{\text{BOUNDARY } \circlearrowleft}$

$\left(\oint_C \langle P, Q\rangle \cdot d\vec{r}\right) \quad \left(\iint_D \text{CURL}\,\vec{F}\cdot\hat{k}\,dA\right)$

$\left(\oint_C \vec{F}\cdot\vec{T}\,ds\right)$

STOKES' THM (3D) (§13.8)

→ II $\displaystyle\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{CURL}\,\vec{F}\cdot d\vec{S}$

$\underbrace{\quad C \qquad\qquad S \quad}_{\text{BOUNDARY}}$

ORIENTED

NORMAL FORM OF GREEN'S THM (§13.5)

III $\displaystyle\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \text{DIV}\,\vec{F}\,dA$

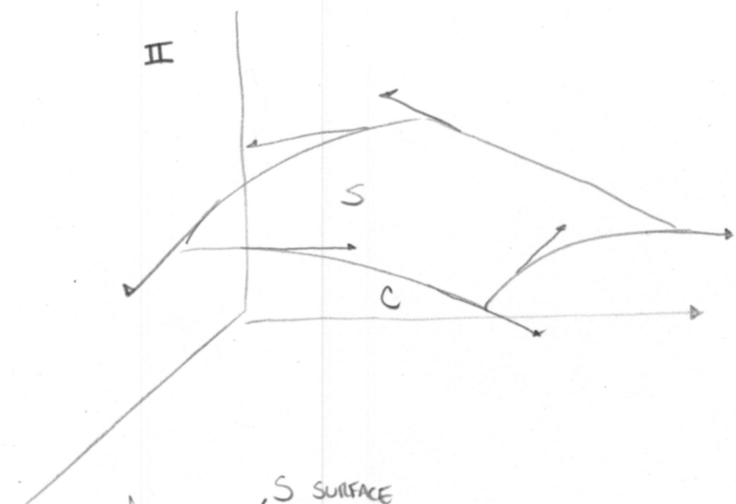
$\underbrace{\quad C \qquad\qquad D \quad}_{\text{BOUNDARY}}$

$\circlearrowleft$

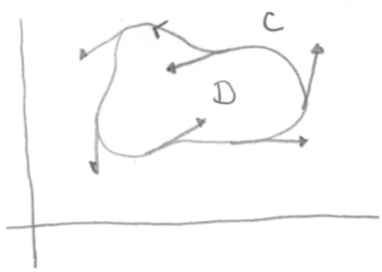
→ DIVERGENCE THM

IV

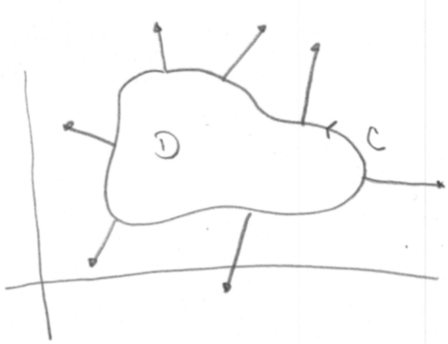
II



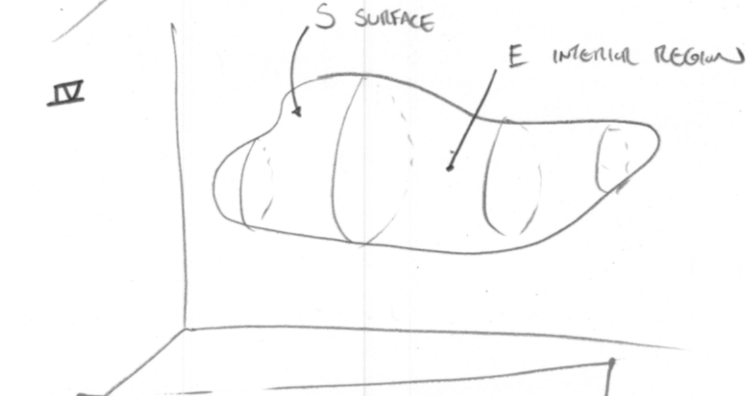
I



III



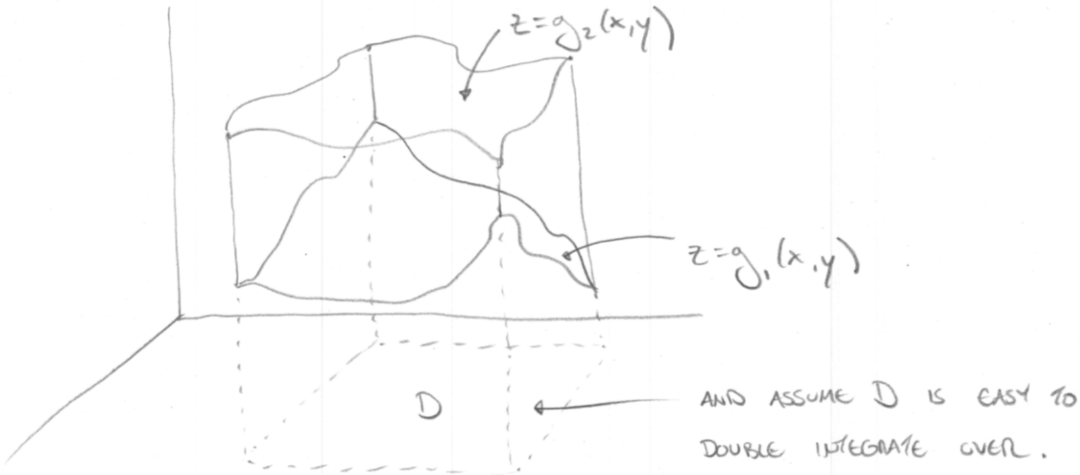
IV



$$\boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{DIV}\,\vec{F}\,dV}$$

RECALL: SOLID REGION OF TYPE 1 IS BOUNDED ABOVE & BELOW BY GRAPHS,

i.e. $\{(x,y,z) : (x,y) \in D,\ g_1(x,y) \le z \le g_2(x,y)\}$



... TYPE 2 : $\{(x,y,z) : (x,z) \in D,\ g_1(x,z) \le y \le g_2(x,z)\}$

... TYPE 3 : $\{(x,y,z) : (y,z) \in D,\ g_1(y,z) \le x \le g_2(y,z)\}$

WE CAN TAKE TRIPLE INTEGRALS OVER THESE SOLID REGIONS.

Def: A SIMPLE SOLID REGION IS A SOLID REGION THAT IS SIMULTANEOUSLY OF TYPE 1, 2, & 3.

DIV. THM

E IS SIMPLE SOLID REGION, $S = \partial E$ WITH OUTWARD ORIENTATION ($\vec{n}$ POINTS OUT).

$\vec{F}$ IS VECTOR FIELD OVER OPEN REGION CONTAINING E, WITH CONTINUOUS PARTIAL DERIVATIVES.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F} \, dV$$

PROOF: $\vec{F} = \langle P, Q, R \rangle$

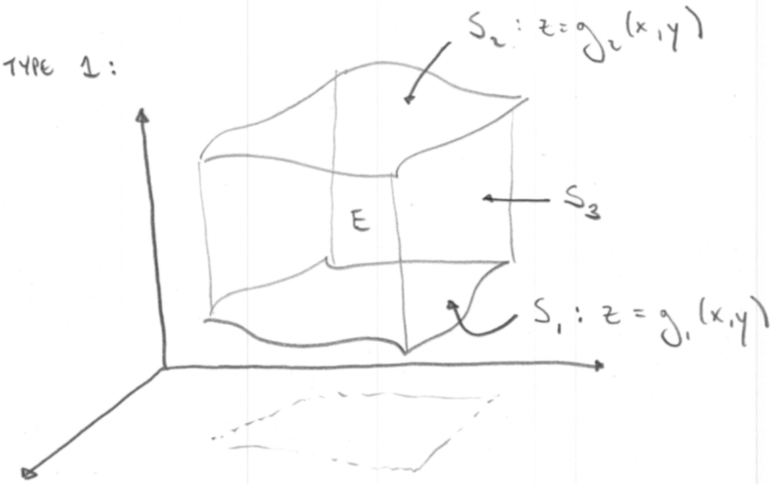
RHS: $$\iiint_E \operatorname{div} \vec{F}\, dV = \iiint_E P_x \, dV + \iiint_E Q_y \, dV + \iiint_E R_z \, dV \quad (1)$$

*

SHOW THESE ARE EQUAL

LHS: $$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \vec{n}\, dS = \iint_S P\hat{i} \cdot \vec{n}\, dS + \iint_S Q\hat{j} \cdot \vec{n}\, dS + \iint_S R\hat{k} \cdot \vec{n}\, dS. \quad (2)$$

* E IS TYPE 1:



(1) $$\iiint_E R_z \, dV = \iint_D \left[\int_{g_1(x,y)}^{g_2(x,y)} R_z \, dz \right] dA = \iint_D \Big(R(x,y,g_2(x,y)) - R(x,y,g_1(x,y))\Big) dA$$

(2) $$\iint_S R\hat{k} \cdot \vec{n}\, dS = \iint_{S_1} R\hat{k} \cdot \vec{n}\, dS + \iint_{S_2} R\hat{k} \cdot \vec{n}\, dS + \iint_{S_3} R\underbrace{\hat{k} \cdot \vec{n}}_{0}\, dS$$

$\hat{k} \perp \vec{n}$ ON S_3 (SIDES)

$$\iint_{S_2} R\hat{k} \cdot \vec{n}\, dS = \iint_D \langle 0,0,R \rangle \cdot \langle -g_{2_x}, -g_{2_y}, 1 \rangle \, dA$$

$$= \iint_D R(x,y,g_2(x,y))\, dA$$

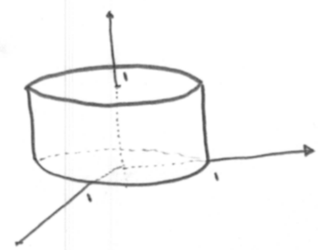
$$\iint_{S_1} R\hat{k}\cdot\vec{n}\,dS = \underset{\text{DOWNWARD NORMAL}}{-}\iint_D \langle 0,0,R\rangle\cdot\langle -g_x, -g_y, 1\rangle\, dA$$

$$= -\iint_D R(x,y,g_1(x,y))\,dA$$

$\therefore$ ② $$\iint_S R\hat{k}\cdot\vec{n}\,dS = \iint_D \Big(R(x,y,g_2(x,y)) - R(x,y,g_1(x,y)) \Big)\, dA$$

SO ① = ②. THE OTHER EQUALITIES ARE SIMILARLY SHOWN. $\square$

ex. $\boxed{2}$ LET S BE THE BOUNDARY OF THE SOLID CYLINDER $\{(x,y,z):\ x^2+y^2 \le 1,\ 0 \le z \le 1\}$, $\vec{F} = \langle xy, yz, zx\rangle$. FIND $\iint_S \vec{F}\cdot d\vec{S}$.



(3 SEPARATE SURFACES)

$$= \iiint_E \operatorname{DIV}\vec{F}\,dV = \iiint_E (y+z+x)\,dV$$

YOU CAN DO THIS DIRECTLY !!

CYLINDRICAL COORD. $x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\rightsquigarrow \int_0^{2\pi}\int_0^1\int_0^1 (r\sin\theta + r\cos\theta + z)\, dz\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r\sin\theta + r\cos\theta + \frac{1}{2}\right) r\,dr\,d\theta = \int_0^{2\pi} \underbrace{\frac{1}{3}(\sin\theta + \cos\theta)}_{0} + \frac{1}{4}\,d\theta = \boxed{\frac{\pi}{2}}$$

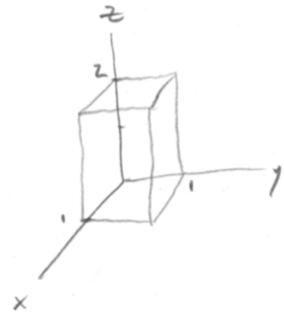
ex. $\boxed{1}$

LET $\vec{F} = \langle e^x \sin y \,,\; e^x \cos y \,,\; yz^2 \rangle$

AND LET S BE BOUNDARY OF SOLID BOX

$\{(x,y,z) : 0 \le x \le 1 \,,\; 0 \le y \le 1 \,,\; 0 \le z \le 2\}$

FIND $\iint\limits_S \vec{F} \cdot d\vec{S}$



$$= \iiint\limits_E \operatorname{DIV} \vec{F} \, dV = \int_0^1 \int_0^1 \int_0^2 \underbrace{e^x \sin y - e^x \sin y}_{0} + 2yz \; dz\, dy\, dx$$

$$= \int_0^1 \int_0^1 4y \; dy\, dx = \int_0^1 2 \, dx = \boxed{2}$$

ex. $\boxed{3}$

LET $\vec{F} = \langle 4x^3 z \,,\; 4y^3 z \,,\; 3z^4 \rangle$ & S BE SPHERE OF RADIUS R

CENTERED AT ORIGIN. FIND $\iint\limits_S \vec{F} \cdot d\vec{S}$

$$= \iiint\limits_E \operatorname{DIV} \vec{F} \, dV = \iiint\limits_E \left(12x^2 z + 12y^2 z + 12z^3 \right) dV$$

SPHERICAL COORD.

$x = \rho \sin\phi \cos\theta$ $\qquad 0 \le \rho \le R$

$y = \rho \sin\phi \sin\theta$ $\qquad 0 \le \theta \le 2\pi$

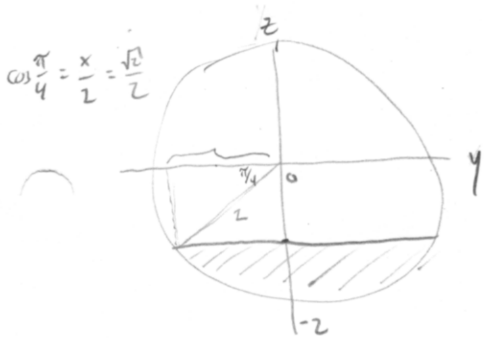
$z = \rho \cos\phi$ $\qquad 0 \le \phi \le \pi$

$$\operatorname{DIV} \vec{F} = 12z\,(x^2 + y^2 + z^2) = 12\rho\cos\phi\,(\rho^2) = 12\rho^3 \cos\phi$$

$$\ldots = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} \left(12\rho^3\cos\phi\right)\rho^2\sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 2R^6 \underbrace{\int_0^{2\pi}\int_0^{\pi} \cos\phi \sin\phi \, d\phi \, d\theta}_{0} = \boxed{0}$$

LET E BE PART OF SOLID BALL $x^2+y^2+z^2 \le 4$ THAT LIES BELOW PLANE $z=-1$



LET $\vec{F} = \langle x, y, z\rangle$.

USE OUTWARD POINTING $\vec{n}$ TO FIND $\iint_{\partial E} \vec{F}\cdot\vec{n}\, dS$

(YOU CAN DO THIS DIRECTLY!)

$$= \iiint_E \operatorname{DIV}\vec{F}\, dV = 3\iiint_E dV = 3\int_0^{2\pi}\int_0^{\sqrt{2}}\int_{-\sqrt{2-r^2}}^{-1} r\, dz\, dr\, d\theta$$

$$= 6\pi\int_0^{\sqrt{2}} (-1 + \underbrace{\sqrt{2-r^2}}_{\substack{u=2-r^2 \\ du=-2r\,dr}})\, r\, dr = 6\pi\left[\int_0^{\sqrt{2}} -r\, dr - \frac{1}{2}\int_2^0 \sqrt{u}\, du\right]$$

$$= 6\pi\left[-1 - \frac{1}{3}u^{3/2}\Big|_2^0\right] = 6\pi\left[-1 + \frac{2\sqrt{2}}{3}\right]$$