CH 4 DETERMINANT

LAPLACE EXPANSION : GIVEN AN $n \times n$ MATRIX $A = (a_{ij})$

AND ANY INTEGER k, $1 \le k \le n$

$$\text{Det } A = \sum_{j=1}^{n} (-1)^{k+j} M_{kj}$$

LAPLACE EXPANSION ACROSS k^{TH} ROW

$$= \sum_{i=1}^{n} (-1)^{i+k} M_{ik}$$

LAPLACE EXPANSION DOWN k^{TH} COLUMN

EX. FIND

$$\left|\begin{matrix} 0 & 0 & 7 & 0 & 2 \\ -12 & -3 & 8 & 2 & -3 \\ 5 & 0 & 4 & 0 & 1 \\ -3 & -4 & 11 & 0 & 5 \\ 0 & 0 & 3 & 0 & 6 \end{matrix}\right| = 2 \left|\begin{matrix} 0 & 0 & 7 & 2 \\ 5 & 0 & 4 & 1 \\ -3 & -4 & 11 & 5 \\ 0 & 0 & 3 & 6 \end{matrix}\right| = 2(-4)(-1) \left|\begin{matrix} 0 & 7 & 2 \\ 5 & 4 & 1 \\ 0 & 3 & 6 \end{matrix}\right|$$

$$= 2(-4)(-1)(5)(-1) \left|\begin{matrix} 7 & 2 \\ 3 & 6 \end{matrix}\right|$$

$$= -40\,(42-6) = -40 \cdot 36 = \boxed{1440}$$

DEF: A TRIANGULAR MATRIX IS A SQ. MATRIX WITH ALL ENTRIES BELOW (OR ABOVE) THE DIAGONAL EQUAL TO ZERO.

$$\begin{bmatrix} 1 & 7 & 8 & 10 \\ 0 & 2 & -2 & -7 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 9 & 0 & 0 & 0 \\ 1 & 8 & 0 & 0 \\ 3 & -1 & 7 & 0 \\ 5 & -2 & 7 & 6 \end{bmatrix}$$

UPPER TRIANGULAR $\qquad$ LOWER TRIANGULAR

THM: THE DETERMINANT OF A TRIANGULAR MATRIX IS THE PRODUCT OF THE DIAGONAL ENTRIES, i.e.

$$\text{DET } A = \prod_{i=1}^{n} a_{ii} = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}.$$

e.g. $\begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} = -6$

GIVEN ANY MATRIX A, WE CAN APPLY ELEMENTARY ROW OPERATIONS TO PRODUCE A TRIANGULAR MATRIX.

RECALL: IF THE MATRIX A IS AN AUGMENTED MATRIX $[A|\vec{b}]$ CORRESPONDING TO A SYSTEM OF EQUATIONS $A\vec{x} = \vec{b}$, ELEMENTARY ROW OPERATIONS PRODUCE AUGMENTED MATRICES WHOSE CORRESPONDING SYSTEME ARE EQUIVALENT (SAME SOLUTION SET).

WHAT EFFECT DO ELEM. ROW OP.'S HAVE ON DETERMINANT?

THM:
- IF TWO ROWS OF A MATRIX ARE INTERCHANGED $\longrightarrow$ DET CHANGES SIGN
- IF ONE ROW OF A MATRIX IS MULTIPLIED BY r $\longrightarrow$ DET MULTIPLIED BY r
- IF A MULTIPLE OF ONE ROW IS ADDED TO A DIFFERENT ROW $\longrightarrow$ DET IS UNCHANGED.

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e.g. FIND DET $\begin{bmatrix} 3 & 4 & 5 & 6 \\ 100 & 200 & 300 & 400 \\ 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

$$= \begin{vmatrix} 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 1 \\ 3 & 4 & 5 & 6 \\ 100 & 200 & 300 & 400 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 1 \\ 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{vmatrix} (100)$$

$$= \begin{vmatrix} 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 1 \\ 0 & -2 & -4 & -18 \\ 0 & 0 & 0 & -4 \end{vmatrix} (100) = \begin{vmatrix} 1 & 2 & 3 & 8 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & -16 \\ 0 & 0 & 0 & -4 \end{vmatrix} (100) = 0$$

COMBINE METHODS: DET $\begin{bmatrix} 3 & 0 & 4 & 2 \\ 1 & -1 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$ = DET $\begin{bmatrix} 0 & 3 & 4 & -1 \\ 1 & -1 & 0 & 1 \\ 0 & -2 & 0 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$

$$= 1(-1) \begin{vmatrix} 3 & 4 & -1 \\ -2 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = (-1) \begin{vmatrix} -1 & 0 & 3 \\ -2 & 0 & -1 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & 3 \\ -2 & -1 \end{vmatrix} = 1 + 6 = \boxed{7}$$

PROPERTIES OF DETERMINANTS

- IF A HAS A ZERO ROW OR ZERO COLUMN, DET $A = 0$

  (YOU CAN MAKE A TRIANGULAR WHILE PRESERVING THE ZERO ROW/COLUMN)

- DET A^T = DET A

- IF ANY ROW (COLUMN) OF A IS A MULTIPLE OF ANOTHER ROW (COLUMN), DET $A = 0$

- DET (AB) = DET(A) DET(B).

e.g. SUPPOSE A IS 4×4 MATRIX WITH DET -2.

FIND ~~DET~~ $(3A^3 A^T)$

$= 3^4 \cdot (-2)^3 (-2) = 81 \cdot 16 = \boxed{1296}$