CH. 5 : INVERSE OF A MATRIX

2 NUMBERS a, b ARE INVERSES OF EACHOTHER IF

$$ab = ba = 1 \qquad (\text{WRITE } b = a^{-1})$$

2 $n \times n$ MATRICES ARE INVERSES OF EACHOTHER IF

$$AB = BA = I_n \qquad (\text{WRITE } B = A^{-1})$$

e.g. $\begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$ AND $\begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$ ARE INVERSES OF EACHOTHER

CHECK: $AB = I_2$

$BA = I_2$ (DON'T NEED TO CHECK BOTH)

NOT EVERY SQ. MATRIX HAS AN INVERSE (RECALL 0 HAS NO INVERSE)

THM: THE $n \times n$ MATRIX A HAS AN INVERSE A^{-1} IF & ONLY IF $\det A \neq 0$.

DEF: IF $\det A = 0$, A IS <u>SINGULAR</u> (<u>NON-INVERTIBLE</u>)

IF $\det A \neq 0$, A IS <u>NONSINGULAR</u>. (<u>INVERTIBLE</u>)

FINDING THE INVERSE : ROW REDUCTION METHOD

THM: IF $\det A \neq 0$, THE R.R.E.F. OF THE AUG. MATRIX $[A|I_n]$ IS THE MATRIX $[I_n|A^{-1}]$.

(NOTE: IF $\det A \neq 0$ THEN RREF OF A IS I_n $\Rightarrow$ A HAS RANK n.)

e.g. FIND A^{-1} WHEN $A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{bmatrix}$

NOTE THAT $\det(A) = 4 \neq 0$

$$[A|I_n] = \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ -1 & 1 & -2 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 & 1 & 0 \\ 0 & 3 & -7 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 3 & -7 & -2 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & -4 & 1 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & -1 & 0 \\ 0 & 0 & 1 & -1/4 & -3/4 & -1/4 \end{array}\right] \sim \begin{array}{cccccc} 1 & -2 & 0 & 7/4 & 9/4 & 3/4 \\ 0 & 1 & 0 & -5/4 & -7/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -3/4 & -1/4 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3/4 & -5/4 & 1/4 \\ 0 & 1 & 0 & -5/4 & -7/4 & -1/4 \\ 0 & 0 & 1 & -1/4 & -3/4 & -1/4 \end{array}\right] = [I_n|A^{-1}]$$

FINDING THE INVERSE : ADJOINT MATRIX METHOD

DEF: GIVEN $n \times n$ MATRIX A, THE ADJOINT OF A, $\operatorname{ADJ} A$, IS CONSTRUCTED BY

1. FINDING ALL n^2 COFACTORS
2. PUTTING THEM INTO A MATRIX OF COFACTORS
3. TAKING THE TRANSPOSE.

i.e. IF $A_{ij} = (i,j)$ COFACTOR $\left(= (-1)^{i+j} M_{ij} \text{ (MINOR)} \right)$

THEN $$\operatorname{ADJ} A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & \ddots & & \vdots \\ \vdots & & & \\ A_{n1} & \cdots & & A_{nn} \end{bmatrix}^T = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & & \ddots & \vdots \\ \vdots & & & \\ A_{1n} & & \cdots & A_{nn} \end{bmatrix}$$

THM: $$A^{-1} = \frac{1}{\operatorname{DET} A} \operatorname{ADJ} A$$

e.g. $$\begin{bmatrix} -8 & 2 \\ 11 & -3 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} -3 & -11 \\ -2 & -8 \end{bmatrix}^T = \frac{1}{2} \begin{bmatrix} -3 & -2 \\ -11 & -8 \end{bmatrix}$$

e.g. IF $A = \begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{bmatrix}$ THEN $\operatorname{DET} A = 4$ SO

$$A^{-1} = \frac{1}{4} \begin{bmatrix} -3 & -5 & -1 \\ -5 & -7 & -3 \\ 1 & -1 & -1 \end{bmatrix}^T = \frac{1}{4} \begin{bmatrix} -3 & -5 & 1 \\ -5 & -7 & -1 \\ -1 & -3 & -1 \end{bmatrix}$$

Q: WHAT IS DETERMINANT OF A^{-1}

USING INVERSE MATRIX TO SOLVE SYSTEMS OF LINEAR EQ'S

ex. SOLVE

$$\begin{aligned} x - 2y + 3z &= 1 \\ -x + y - 2z &= -2 \\ 2x - y - z &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \longleftrightarrow A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} -3 & -5 & 1 \\ -5 & -7 & -1 \\ -1 & -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 8 \\ 8 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$$

$x = 2$, $y = 2$, $z = 1$.

* NOTE: THIS ONLY WORKS WHEN A IS INVERTIBLE $\longleftrightarrow$ NON-SINGULAR ($\det \neq 0$) $\longleftrightarrow$ RANK n

$\longleftrightarrow$ UNIQUE SOLUTION.

$\downarrow$

IN THIS CASE, WE ALSO HAVE

CRAMER'S RULE: SUPPOSE $A\vec{x} = \vec{b}$, A NON-SINGULAR. LET N_i BE THE $n \times n$ MATRIX FORMED BY REPLACING THE i^{th} COLUMN OF A WITH $\vec{b}$.

THEN $$x_i = \frac{\det N_i}{\det A} \quad \text{FOR } i = 1, \ldots, n.$$

ex.

$$\begin{aligned} x \quad\quad\; + z &= 1 \\ 2x + y + z &= 1 \\ 3x + 2y + 2z &= 1 \end{aligned} \quad \longleftrightarrow \quad \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

THEN

$$x = \frac{\text{DET} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 2 \end{bmatrix}}{\text{DET} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}}, \quad y = \frac{\text{DET} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}}{\text{DET} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}},$$

$$z = \frac{\text{DET} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}}{\text{DET} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}}.$$

PROPERTIES OF INVERSE

SUPPOSE A, B NONSINGULAR $n \times n$ MATRICES & c IS REAL NON-ZERO NUMBER

- $(A^{-1})^{-1} = A$
- $I_n^{-1} = I_n$
- $(A^T)^{-1} = (A^{-1})^T$
- $(AB)^{-1} = B^{-1} A^{-1}$
- $(cA)^{-1} = c^{-1} A^{-1}$