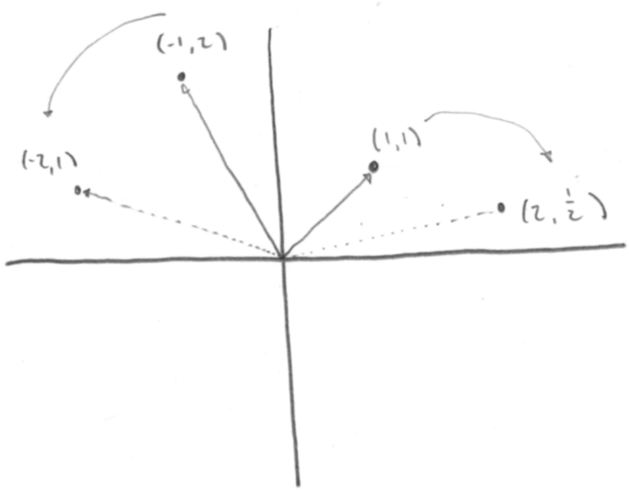
CH. 7 THE EIGENVALUE PROBLEM

CONSIDER $A = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, AND HOW IT ACTS ON (COLUMN) VECTORS IN $\mathbb{R}^2$.

$$A\vec{x} = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ \frac{1}{2}y \end{bmatrix}$$

SINCE THIS IS ARBITRARY, WE SEE WHAT A DOES TO ALL VECTORS IN $\mathbb{R}^2$.

ASSOCIATING VECTORS WITH THEIR TERMINAL POINTS, WE SAY A IS A LINEAR TRANSFORMATION OF $\mathbb{R}^2$



THE PLANE GETS STRETCHED HORIZONTALLY BY FACTOR 2 & COMPRESSED VERTICALLY BY FACTOR $\frac{1}{2}$.

(SQUARES $\longmapsto$ PARALLELOGRAMS)

IN GENERAL, VECTORS GET MAPPED TO OTHER VECTORS WITH DIFFERENT MAGNITUDES & DIRECTIONS

BUT NOT ALL!

THE ACTION OF A ON THESE VECTORS IS THAT DIRECTION IS INVARIANT. ONLY MAGNITUDE CHANGES.

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ 0 \end{bmatrix} = \begin{bmatrix} 2a \\ 0 \end{bmatrix} \qquad A\vec{a} = 2\vec{a}}_{(i)}$$

$$\underbrace{\begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{b}{2} \end{bmatrix} \qquad A\vec{b} = \frac{1}{2}\vec{b}}_{(ii)}$$

IN (i) WE SAY 2 IS AN EIGENVALUE & $\vec{a}$ IS AN EIGENVECTOR ASSOCIATED WITH THE EIGENVALUE 2.

IN (ii) .... $\frac{1}{2}$ EIGENVALUE ... $\vec{b}$ EIGENVECTOR.

NOTE: THE EIGENVECTORS ARE NOT UNIQUE. GIVEN ONE, ANY MULTIPLE OF IT IS ALSO AN EIGENVECTOR.

__DEF:__ λ IS AN __EIGENVALUE__ OF AN $n \times n$ MATRIX A IF THE __EIGENVECTOR EQUATION__ $A\vec{x} = \lambda \vec{x}$, WHERE $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ HAS NON-TRIVIAL SOLUTIONS $\vec{x}$ ($\vec{x} = \vec{0}$ DOESN'T COUNT!).

IF λ IS AN EIGENVALUE OF A THE EVERY SOLUTION $\vec{x}$ OF $A\vec{x} = \lambda\vec{x}$ IS CALLED AN EIGENVECTOR OF A CORRESPONDING TO λ.

__FINDING EIGENVALUES:__

$$A\vec{x} = \lambda \vec{x}$$

$$A\vec{x} - \lambda\vec{x} = 0$$

$$A\vec{x} - \lambda I \vec{x} = 0$$

$$(*) \quad (A - \lambda I)\vec{x} = 0$$

"HOMOGENEOUS"

NOTE THAT $\vec{x} = 0$ IS A (TRIVIAL) SOLUTION.
WE WANT NON-TRIVIAL SOLUTIONS.
RECALL THAT IF $\det(A - \lambda I) \neq 0$ THEN THE SOLUTION IS UNIQUE.

SO WE NEED $\det(A - \lambda I) = 0$.

WE FOCUS ON CASE $n = 2$.

e.g. $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

THEN $\operatorname{Det}(A - \lambda I) = \left| \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix} \quad \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$

$$= \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2$$

$$= \lambda^2 - 5\lambda + 4$$

SOLVE $\operatorname{DET}(A - \lambda I) = 0 \implies \overbrace{\lambda^2 - 5\lambda + 4}^{\text{CHARACTERISTIC POLYNOMIAL}} = 0$

$$(\lambda - 4)(\lambda - 1) = 0$$

EIGENVALUES: $\lambda = 4 \qquad \lambda = 1$

PRACTICE: FIND EIGENVALUES FOR $\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$. $\quad (\lambda = 6,\ \lambda = 1)$

FINDING CORRESPONDING EIGENVECTORS (REMEMBER: NOT UNIQUE (SCALAR MULTIPLES WORK TOO!))

e.g. $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$, $\lambda = 4, 1$

SET $\lambda = 4$. SOLVE FOR $\vec{x}$ $\quad (A - \lambda I)\vec{x} = 0$

$$\begin{bmatrix} 2-4 & 2 \\ 1 & 3-4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ROW OP $\longrightarrow$ $\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ i.e. $x - y = 0$

$x = y$

LET $y = r$. $\Rightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r \\ r \end{bmatrix} = \boxed{r \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ FOR ANY } r \in \mathbb{R}.}$

THEN $x = r$

EIGENVECTOR

FAMILY OF EIGENVECTORS CORRESPONDING TO EIGENVALUE $\lambda = 4$.

SIMILARLY, FOR $\lambda = 1$: $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\Rightarrow$ $x = -2y$

LET $y = s$

THEN $x = -2s$

$\therefore \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2s \\ s \end{bmatrix} = \boxed{s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ FOR ANY } s \in \mathbb{R}}$

<u>PRACTICE</u>: FIND EIGENVECTORS FOR $\begin{bmatrix} 5 & -2 \\ -2 & 2 \end{bmatrix}$ WITH $\lambda = 6, 1$

$\left(\lambda = 6 \rightarrow r \begin{bmatrix} -2 \\ 1 \end{bmatrix}, r \in \mathbb{R}. \quad \lambda = 1 \rightarrow s \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$

APPLICATION TO DIFF. EQ.'S

e.g. SOLVE THE SYSTEM OF O.D.E.'S

$$y_1' = 2y_1 + 2y_2$$
$$y_2' = y_1 + 3y_2$$

WITH INITIAL COND.

$$y_1(0) = 2$$
$$y_2(0) = 1$$

LET $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ AND $\vec{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix}$.

THEN WE HAVE $\vec{y}' = A\vec{y}$.

NOTE THAT IN CASE $n=1$ THIS IS:

$y' = Ay$ WITH SOLUTION $y(t) = e^{At}c$ (CHECK!)

CONSTANT DETERMINED BY INITIAL CONDITION

TRY SETTING $\vec{y} = e^{\lambda t}\vec{c}$ i.e. $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda t}c_1 \\ e^{\lambda t}c_2 \end{bmatrix}$

AND PLUGGING INTO $\vec{y}' = A\vec{y}$

$$\rightarrow \lambda \cancel{e^{\lambda t}}\vec{c} = A\cancel{e^{\lambda t}}\vec{c}$$

$$\lambda\vec{c} = A\vec{c}$$

λ IS EIGENVALUE OF A WITH CORRESPONDING EIGEN-VECTOR $\vec{c}$

$\lambda=4$: $\vec{y}_I = e^{4t}r\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{y}_I = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} re^{4t} \\ re^{4t} \end{bmatrix}$

$\lambda=1$: $\vec{y}_{II} = e^{t}s\begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow \vec{y}_{II} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} -2se^{t} \\ se^{t} \end{bmatrix}$

SO TWO SOLUTIONS: $\left.\begin{aligned}\vec{y}_I\,' &= A\vec{y}_I \\ \vec{y}_{II}\,' &= A\vec{y}_{II}\end{aligned}\right\} \Rightarrow \vec{y}_I\,' + \vec{y}_{II}\,' = A\vec{y}_I + A\vec{y}_{II}$

$$(\vec{y}_I + \vec{y}_{II})' = A(\vec{y}_I + \vec{y}_{II})$$

ADDING THE 2 SOLUTIONS GIVES ANOTHER SOLUTION!

SOLUTION:

$$\vec{y}_I + \vec{y}_{II} = \begin{bmatrix} re^{4t} \\ re^{4t} \end{bmatrix} + \begin{bmatrix} -2se^{t} \\ se^{t} \end{bmatrix} \longrightarrow \underbrace{\begin{aligned} y_1(t) &= re^{4t} - 2se^{t} \\ y_2(t) &= re^{4t} + se^{t} \end{aligned}}_{\text{GENERAL SOLUTION TO O.D.E.}}$$

$$y_1' = 2y_1 + 2y_2$$
$$y_2' = y_1 + 3y_2$$

<u>INITIAL CONDITIONS:</u> $\begin{aligned} y_1(0) &= 2 \\ y_2(0) &= 1 \end{aligned}$

$$\begin{aligned} y_1(0) &= r - 2s = 2 \\ y_2(0) &= r + s = 1 \end{aligned} \longrightarrow \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 3 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & -1/3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 4/3 \\ 0 & 1 & -1/3 \end{array}\right] \Rightarrow \begin{aligned} r &= 4/3 \\ s &= -1/3 \end{aligned}$$

$$\therefore \boxed{\begin{aligned} y_1(t) &= \frac{4}{3}e^{4t} + \frac{2}{3}e^{t} \\ y_2(t) &= \frac{4}{3}e^{4t} - \frac{1}{3}e^{t} \end{aligned}} \quad (\text{CHECK?})$$

PRACTICE:

$$\begin{cases} y_1' = -5y_1 + 2y_2 & y_1(0) = 1 \\ y_2' = 2y_1 - 2y_2 & y_2(0) = 2 \end{cases}$$

$$\begin{cases} y_1' = 2y_1 - y_2 & y_1(0) = 2 \\ y_2' = 2y_1 + 5y_2 & y_2(0) = -1 \end{cases}$$