THE CITY COLLEGE OF NEW YORK
DEPARTMENT OF MATHEMATICS
SPRING 2017

MATH 392, FINAL EXAMINATION

YOUR NAME (print and sign):

* Answer Key *

NAME OF YOUR INSTRUCTOR:

Pr. 1	Pr. 2	Pr. 3	Pr. 4	Pr. 5	Pr. 6	Pr. 7	Pr. 8	Pr. 9	Pr. 10	Pr. 11	Total

INSTRUCTIONS:

- There are a total of 11 problems.
- DO ALL PROBLEMS 1 THROUGH 7 AND THREE OF THE FOUR PROBLEMS 8-11. IN THE TABLE ABOVE, CROSS OUT ONE PROBLEM AMONG PROBLEMS 8-11 THAT YOU OMITTED.
- Each problem is worth 10 points.
- Notes, books and calculators are not to be used.
- All work on this exam is to be your own.
- Read each problem carefully. Be sure to show your work. Remember that it is your obligation to answer each question clearly and in a way that convinces the grader that you understand how to solve the problem.
- Stop working immediately at the end of the exam when time is called.

1. Find the line integral

$$\int_C y^3 ds,$$

where C is the curve given by

$$\vec{r}(t) = (1, t, t^2/2), \quad 0 \le t \le 1.$$

$$\vec{r}\,'(t) = \langle 0, 1, t \rangle$$

$$|\vec{r}\,'(t)| = \sqrt{1 + t^2}$$

$$\int_C f\, ds = \int_0^1 f(\vec{r}(t))\,|\vec{r}\,'(t)|\, dt$$

$$= \int_0^1 t^3 \sqrt{1 + t^2}\, dt$$

LET $u = 1 + t^2$

$du = 2t\, dt$

Note : $t^3 dt = \frac{1}{2}(u-1)\, du$

$$\rightsquigarrow \frac{1}{2}\int_1^2 (u-1)\sqrt{u}\, du = \frac{1}{2}\int_1^2 \left(u^{3/2} - u^{1/2}\right) du$$

$$= \frac{1}{2}\left[\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right]_1^2$$

$$= \frac{1}{5}\left(2^{5/2} - 1\right) - \frac{1}{3}\left(2^{3/2} - 1\right) = \boxed{\frac{2(\sqrt{2}+1)}{15}}$$

2. Consider the following system of linear equations.

$$\begin{cases} x_1 - 2x_2 + x_3 - x_4 = 1, \\ x_1 - x_3 + x_4 = 3, \\ -x_2 + x_3 - x_4 = -1. \end{cases}$$

Determine whether this system is consistent, and if it is, find the full set of solutions. Also, find the rank of the matrix of coefficients.

AUG. MATRIX :

$$\begin{bmatrix} 1 & -2 & 1 & -1 & 1 \\ 1 & 0 & -1 & 1 & 3 \\ 0 & -1 & 1 & -1 & -1 \end{bmatrix} \quad R_2 - R_1 \quad \begin{bmatrix} 1 & -2 & 1 & -1 & 1 \\ 0 & 2 & -2 & 2 & 2 \\ 0 & -1 & 1 & -1 & -1 \end{bmatrix}$$

$$\tfrac{1}{2}R_2 \quad \begin{bmatrix} 1 & -2 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & -1 & 1 & -1 & -1 \end{bmatrix} \quad \begin{matrix} R_1 + 2R_2 \\ \\ R_3 + R_2 \end{matrix} \quad \begin{bmatrix} 1 & 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

THE SYSTEM IS $\boxed{\text{CONSISTENT.}}$

THE RANK OF COEFF. MATRIX IS $\boxed{2.}$

$$x_1 - x_3 + x_4 = 3 \longrightarrow x_1 = 3 + x_3 - x_4 \qquad \text{LET } x_3 = \alpha \in \mathbb{R}$$

$$x_2 - x_3 + x_4 = 1 \longrightarrow x_2 = 1 + x_3 - x_4 \qquad x_4 = \beta \in \mathbb{R}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 + \alpha - \beta \\ 1 + \alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \boxed{\begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} ; \ \alpha, \beta \in \mathbb{R}}$$

3. Consider the following vector field.

$$\vec{F}(x,y,z) = x\vec{i} + \frac{y}{1+y^2+z^2}\vec{j} + \frac{z}{1+y^2+z^2}\vec{k}.$$

(a) Find curl$\vec{F}$.
(b) Is $\vec{F}$ conservative? Make sure to justify your claim.

(a) $\text{CURL } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & \frac{y}{1+y^2+z^2} & \frac{z}{1+y^2+z^2} \end{vmatrix} = \left\langle \frac{-2yz+2yz}{(1+y^2+z^2)^2}, 0, 0 \right\rangle = \boxed{\vec{0}}$

$\boxed{\text{YES}}$,

(b) $\vec{F}$ IS CONSERVATIVE BECAUSE $\boxed{\text{DOM}(\vec{F}) = \mathbb{R}^3}$ (SIMPLY CONNECTED).

AND $\boxed{\text{CURL } \vec{F} = \vec{0}.}$

4. Find the work done by the vector field

$$\vec{F}(x,y) = y^2\vec{\imath} - xe^y\vec{\jmath}$$

moving a particle along the curve C given by

$$\vec{r}(t) = (t,t), \quad 0 \le t \le 1.$$

$$\vec{r}\,'(t) = \langle 1, 1 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 \langle t^2, -te^t \rangle \cdot \langle 1, 1 \rangle \, dt$$

$$= \int_0^1 (t^2 - te^t)\, dt = \frac{1}{3} t^3 \Big|_0^1 - \int_0^1 te^t\, dt$$

$u = t \qquad v = e^t$

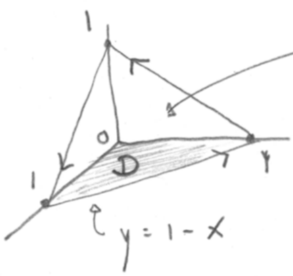
$du = dt \qquad dv = e^t\, dt$

$$= \frac{1}{3} - te^t \Big|_0^1 + \int_0^1 e^t\, dt$$

$$= \frac{1}{3} - e + e^t \Big|_0^1 = \frac{1}{3} - e + (e - 1)$$

$$= \boxed{-\frac{2}{3}}$$

5. Use Stokes' Theorem to find
$$\int_C xydx - yzdz,$$
where C is the boundary curve of
$$\{(x, y, z) : \ x + y + z = 1, \ x, y, z \geq 0\},$$
oriented clockwise as seen from the origin.



$S: \vec{r}(x,y) = \langle x, y, 1-x-y \rangle$

$\vec{r}_x \times \vec{r}_y = \langle 1, 1, 1 \rangle \qquad \left.\begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1-x \end{array}\right\} D$

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 0 & -yz \end{vmatrix} = \langle -z, 0, -x \rangle$$

$x + y - 1$

STOKES' THM: $\displaystyle\oint_C \vec{F} \cdot d\vec{r} = \iint_S \text{curl}\,\vec{F} \cdot d\vec{S} = \iint_D \langle x+y-1, 0, -x \rangle \cdot \langle 1, 1, 1 \rangle \, dA$

$$= \int_0^1 \int_0^{1-x} (y-1)\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2 - y\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \frac{1}{2}(1-x)^2 - (1-x)\,dx = \int_0^1 \left(\frac{1}{2}x^2 - \frac{1}{2}\right) dx$$

$$\frac{1}{2} - \cancel{x} + \frac{1}{2}x^2 - 1 + \cancel{x}$$

$$= \frac{1}{6}x^3 - \frac{1}{2}x \Big|_0^1 = \frac{1}{6} - \frac{1}{2} = \boxed{-\frac{1}{3}}$$

6. Use the Fundamental Theorem for Line Integrals to find

$$\int_C y\cos(xy)dx + (x\cos(xy) - ze^{yz})dy - ye^{yz}dz,$$

where C is the curve given by

$$\vec{r}(t) = (t, \pi t/2, 1-t), \quad 0 \le t \le 1.$$

$$\nabla f = \langle y\cos(xy),\ x\cos(xy) - ze^{yz},\ ye^{yz} \rangle$$

$$f(x,y,z) = \sin(xy) + g(y,z)$$

$$\Rightarrow\ f_y = x\cos(xy) + g_y = x\cos(xy) - ze^{yz}$$

$$g(y,z) = -e^{yz} + \underbrace{h(z)}_{0}$$

$$\therefore\ f(x,y,z) = \sin(xy) - e^{yz}$$

FTLI: $\displaystyle\int_C \nabla f \cdot d\vec{r} = f(\vec{r}(1)) - f(\vec{r}(0))$

$$= f(1, \tfrac{\pi}{2}, 0) - f(0,0,1)$$

$$= \left(\sin(\tfrac{\pi}{2}) - 1\right) - (0 - 1)$$

$$= \boxed{1}$$

7. Consider the following matrix.

$$A = \begin{bmatrix} 0 & 2 & -1 \\ 2 & 4 & 1 \\ -3 & 0 & -3 \end{bmatrix}.$$

Find A^{-1} and use it to solve

$$\begin{cases} 2x_2 - x_3 = 0, \\ 2x_1 + 4x_2 + x_3 = 2, \\ -3x_1 - 3x_3 = 3. \end{cases}$$

$$A \mid I : \left[\begin{array}{ccc|ccc} 0 & 2 & -1 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -3 & 0 & -3 & 0 & 0 & 1 \end{array}\right] \quad R_1 \leftrightarrow R_2 \quad \left[\begin{array}{ccc|ccc} 2 & 4 & 1 & 0 & 1 & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ -3 & 0 & -3 & 0 & 0 & 1 \end{array}\right]$$

$$\begin{array}{l} \frac{1}{2} R_1 \\ \\ R_3 + 3R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 2 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 6 & -\frac{3}{2} & 0 & \frac{3}{2} & 1 \end{array}\right] \quad \begin{array}{l} R_1 - R_2 \\ \\ R_3 - 3R_2 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & \frac{3}{2} & -3 & \frac{3}{2} & 1 \end{array}\right]$$

$$\begin{array}{l} \\ \frac{1}{2} R_2 \\ \frac{2}{3} R_3 \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & \frac{3}{2} & -1 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & \frac{2}{3} \end{array}\right] \quad \begin{array}{l} R_1 - \frac{3}{2} R_3 \\ R_2 + \frac{1}{2} R_3 \\ \\ \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -2 & 1 & \frac{2}{3} \end{array}\right]$$

$$\therefore \boxed{A^{-1} = \begin{bmatrix} 2 & -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ -2 & 1 & \frac{2}{3} \end{bmatrix}} ; \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{3} \\ -2 & 1 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -5 \\ 2 \\ 4 \end{bmatrix}}$$

8. Find the area of the surface S given by

$$\vec{r}(u,v) = (uv, -u, v),$$

for $u^2 + v^2 \leq 1$.

$\vec{r}_u = \langle v, -1, 0 \rangle$; $\vec{r}_v = \langle u, 0, 1 \rangle$

$$|\vec{r}_u \times \vec{r}_v| = \left\| \begin{matrix} \hat{i} & \hat{j} & \hat{k} \\ v & -1 & 0 \\ u & 0 & 1 \end{matrix} \right\| = |\langle -1, -v, -u \rangle|$$

$$= \sqrt{1 + v^2 + u^2}$$

$$\iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v| \, dA = \int_0^{2\pi} \int_0^1 \sqrt{1 + r^2} \, r \, dr \, d\theta$$

LET $u = 1 + r^2$

$du = 2r\,dr$

$$= \pi \int_1^2 \sqrt{u} \, du = \frac{2\pi}{3} u^{3/2} \Big|_1^2 = \boxed{\frac{2\pi}{3}\left(2\sqrt{2} - 1\right)}$$

9. (a) Find the eigenvalues and eigenvectors for the matrix

$$A = \begin{bmatrix} -1 & 3 \\ 1 & -1 \end{bmatrix}.$$

(b) Find the general solution to the following system of linear ordinary differential equations.

$$\begin{cases} y_1' = -y_1 + 3y_2, \\ y_2' = y_1 - y_2. \end{cases}$$

(a) $(-1-\lambda)^2 - 3 = 0$

$(-1-\lambda)^2 = 3$

$-1-\lambda = \pm\sqrt{3}$

$\underline{\underline{\lambda = -1 \pm \sqrt{3}}}$

$\lambda = -1+\sqrt{3}$: $\begin{bmatrix} -\sqrt{3} & 3 \\ 1 & -\sqrt{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \rightarrow -\sqrt{3}x + 3y = 0$

$x = \sqrt{3}y$

$r\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}$

$\lambda = -1-\sqrt{3}$: $\begin{bmatrix} \sqrt{3} & 3 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \rightarrow \sqrt{3}x + 3y = 0$

$x = -\sqrt{3}y$

$s\begin{bmatrix} -\sqrt{3} \\ 1 \end{bmatrix}$

(b)

$$y_1(t) = r\sqrt{3}e^{(-1+\sqrt{3})t} - s\sqrt{3}e^{(-1-\sqrt{3})t}$$

$$y_2(t) = re^{(-1+\sqrt{3})t} + se^{(-1+\sqrt{3})t}$$

10. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ -1 & 3 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{bmatrix}.$$

Find $\det(A)$ and $\det\left(\frac{1}{2}AA^TA^{-1}\right)$.

$$\begin{array}{c} \\ \\ R_3+R_1 \\ R_4-3R_1 \end{array}\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 4 & 1 & 2 \\ 0 & -1 & 0 & -5 \end{bmatrix} \quad \begin{array}{c} \\ \\ R_3-4R_2 \\ R_4+R_2 \end{array}\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -10 \\ 0 & 0 & 1 & -2 \end{bmatrix} \quad \begin{array}{c} \\ \\ \\ R_4+\frac{1}{3}R_3 \end{array}\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -3 & -10 \\ 0 & 0 & 0 & -\frac{16}{3} \end{bmatrix}$$

$$\text{DET}(A) = (1)(1)(-3)\left(-\frac{16}{3}\right) = \boxed{16}$$

$$\text{DET}\left(\frac{1}{2}AA^TA^{-1}\right) = \left(\frac{1}{2}\right)^4 \text{DET}(A)\,\text{DET}(A^T)\,\text{DET}(A^{-1})$$

$$= \left(\frac{1}{2}\right)^4 (16)(16)\left(\frac{1}{16}\right) = \boxed{1}$$

11. Use the Divergence Theorem to find the surface integral

$$\iint_S \vec{F} \cdot d\vec{S},$$

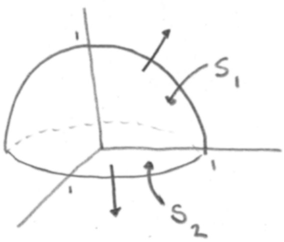
where

$$\vec{F}(x,y,z) = x\vec{\imath} - y\vec{\jmath} + z^2\vec{k},$$

and S is the boundary surface of

$$E = \{(x,y,z):\ 0 \le z \le \sqrt{1-x^2-y^2}\},$$

oriented so that the normal is outward pointing.



$$\text{DIV}\,\vec{F} = 1 - 1 + 2z = 2z$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{DIV}\,\vec{F}\, dV = \iiint_E 2z\, dV$$

SPHERICAL COORD: $z = \rho\cos\phi$, $dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

$$\to 2\int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho^3 \sin\phi\cos\phi\, d\rho\, d\phi\, d\theta$$

$$= \pi\int_0^{\pi/2} \sin\phi\cos\phi\, d\phi = \frac{\pi}{2}\sin^2\phi\,\Big|_0^{\pi/2} = \boxed{\frac{\pi}{2}}$$

ALT: CYLINDRICAL COORD: $dV = r\, dz\, dr\, d\theta$

$$\to 2\int_0^{2\pi}\int_0^1\int_0^{\sqrt{1-r^2}} zr\, dz\, dr\, d\theta = 2\pi\int_0^1 (1-r^2)r\, dr$$

$$= 2\pi\left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 = \frac{\pi}{2}$$