Name: *Answer Key* 4/19/2017

Math 392 Linear Algebra and Vector Analysis for Engineers **Midterm Exam**

1. (8 points) Evaluate the following line integral with respect to arc length, where C is the line segment from $(0,0,0)$ to $(1,2,3)$.

$$\int_C xe^{yz}\, ds$$

$C: \vec{r}(t) = \langle t, 2t, 3t \rangle, \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 1, 2, 3 \rangle$

$|\vec{r}\,'(t)| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$

$$\int_C f\, ds = \int_0^1 f(\vec{r}(t))\, |\vec{r}\,'(t)|\, dt$$

$$= \sqrt{14} \int_0^1 t e^{6t^2}\, dt$$

$u = 6t^2$

$du = 12t$

$$= \frac{\sqrt{14}}{12} \int_0^6 e^u\, du = \boxed{\frac{\sqrt{14}}{12}\left(e^6 - 1\right)}$$

2. Let $\mathbf{F}(x,y) = (ax^2y + y^3 + 1)\vec{i} + (2x^3 + bxy^2 + 2)\vec{j}$ be a vector field, where a and b are constants.

(a) (4 points) Find the values of a and b for which $\mathbf{F}$ is conservative.

(b) (4 points) For these values of a and b, find $f(x,y)$ such that $\mathbf{F} = \nabla f$.

(c) (4 points) Still using the values of a and b from part (a), compute $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C such that $x = e^t \cos t$, $y = e^t \sin t$, $0 \le t \le \pi$.

(a) THE COMPONENTS OF $\vec{F}$ ARE POLYNOMIALS. THUS, THEY ARE DEFINED ON ALL OF $\mathbb{R}^2$ (A SIMPLY CONNECTED DOMAIN) AND HAVE CONTINUOUS PARTIAL DERIVATIVES.

$\therefore \vec{F}$ IS CONSERVATIVE IF $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ (WHERE $\vec{F} = \langle P, Q \rangle$).

$$\frac{\partial Q}{\partial x} = 6x^2 + by^2 \quad \text{AND} \quad \frac{\partial P}{\partial y} = ax^2 + 3y^2$$

SETTING THESE EQUAL, $\boxed{\begin{array}{l} a = 6 \\ b = 3 \end{array}}$

(b) $P = f_x = 6x^2y + y^3 + 1 \Rightarrow f = 2x^3y + xy^3 + x + g(y)$

$\Rightarrow Q = f_y = 2x^3 + 3xy^2 + \underbrace{g'(y)}_{2} \Rightarrow \boxed{f(x,y) = 2x^3y + xy^3 + x + 2y}$

(c) $\vec{r}(t) = \langle e^t \cos t, e^t \sin t \rangle$, $0 \le t \le \pi$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0)) = f(-e^{\pi}, 0) - f(1, 0)$$

$$= \boxed{-e^{\pi} - 1}$$

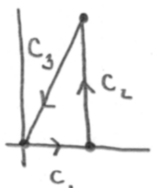
3. Verify that Green's theorem is true for the line integral

$$\int_C xy\,dx + x^2\,dy$$

where C is the triangle with vertices $(0,0), (1,0), (1,2)$ oriented counterclockwise. Do this by calculating

(a) (8 points) the line integral directly, and

(b) (8 points) the related double integral.

(a) 

C_1: $x = t \quad y = 0 \quad , \quad 0 \le t \le 1$

$dx = dt \quad dy = 0\,dt$

$$\int_{C_1} = \int_0^1 (t \cdot 0 + 0)\,dt = \underline{\underline{0}}$$

C_2: $x = 1 \quad y = 2t \quad , \quad 0 \le t \le 1$

$dx = 0\,dt \quad dy = 2\,dt$

$$\int_{C_2} = \int_0^1 (2t \cdot 0 + 2)\,dt = \underline{\underline{2}}$$

$-C_3$: $x = t \quad y = 2t \quad , \quad 0 \le t \le 1$

$dx = dt \quad dy = 2dt$

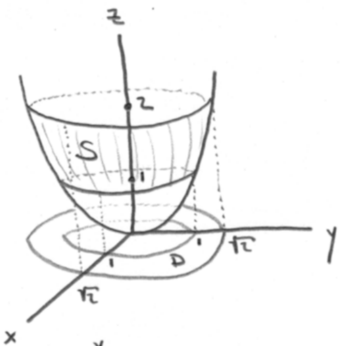
$$\int_{C_3} = -\int_{-C_3} = -\int_0^1 (2t^2 + 2t^2)\,dt = \underline{\underline{-\frac{4}{3}}}$$

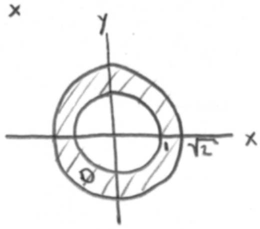
$$\therefore \int_C = \int_{C_1} + \int_{C_2} + \int_{C_3} = 0 + 2 - \frac{4}{3} = \boxed{\frac{2}{3}}$$

(b) $$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_0^1 \int_0^{2x} x\,dy\,dx = \int_0^1 2x^2\,dx = \boxed{\frac{2}{3}}$$

✓

4. (8 points) Find the surface area of the part of the surface $z = x^2 + y^2$ with $1 \leq z \leq 2$.





$S: \vec{r}(x,y) = \langle x, y, x^2+y^2 \rangle, \quad 1 \leq x^2+y^2 \leq 2$

$|\vec{r}_x \times \vec{r}_y| = \sqrt{(-2x)^2 + (-2y)^2 + 1}$

$= \sqrt{4(x^2+y^2)+1}$

$\text{Area} = \iint_S 1\, dS = \iint_D |\vec{r}_x \times \vec{r}_y|\, dA$

$= \int_0^{2\pi} \int_1^{\sqrt{2}} r\sqrt{4r^2+1}\, dr\, d\theta$

$u = 4r^2+1$

$du = 8r\, dr$

$= \frac{\pi}{4} \int_5^9 \sqrt{u}\, du = \frac{\pi}{6} u^{3/2} \Big|_5^9 = \boxed{\frac{\pi}{6}(27 - 5\sqrt{5})}$

PLANE

CYLINDER

5. Let C be the intersection curve of the surfaces

$$z = 3x \quad \text{and} \quad x^2 + y^2 = 1$$

oriented counterclockwise as seen from above. Calculate

INTERSECTION CURVE IS AN ELLIPSE.

$$\int_C (1-4z)\,dx + 2x\,dy + (1-5z)\,dz$$

(a) (8 points) directly as a line integral, and

(b) (8 points) as a double integral, by using Stoke's theorem.

(a) LET $x = \cos t \qquad y = \sin t \qquad z = 3\cos t \qquad 0 \le t \le 2\pi$

$dx = -\sin t \qquad dy = \cos t \qquad dz = -3\sin t$

$$\int_C = \int_0^{2\pi} \Big((1-12\cos t)(-\sin t) + 2\cos^2 t + (1-15\cos t)(-3\sin t) \Big)\, dt$$

$$= \int_0^{2\pi} \left(-\sin t + 12\sin t\cos t + 2\cos^2 t - 3\sin t + 45\sin t\cos t \right) dt$$

$$= \int_0^{2\pi} \underbrace{-4\sin t}_{0} + \underbrace{57\sin t\cos t}_{0} + 2\cos^2 t = 2\int_0^{2\pi} \underbrace{\cos^2 t}_{\frac{1}{2}(1+\cos 2t)}\, dt = \boxed{2\pi}$$

(b) $S: \vec{r}(x,y) = \langle x, y, 3x \rangle, \quad x^2 + y^2 \le 1$

$\vec{r}_x \times \vec{r}_y = \langle -3, 0, 1 \rangle$

$$\text{CURL } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1-4z & 2x & 1-5z \end{vmatrix} = \langle 0, -4, 2 \rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_S \text{CURL } \vec{F}\, dS = \iint_D \text{CURL } \vec{F} \cdot (\vec{r}_x \times \vec{r}_y)\, dA = 2 \underbrace{\iint_D dA}_{\text{AREA OF UNIT CIRCLE}} = \boxed{2\pi}$$