**Name:** *ANSWER KEY* **3/29/2017**

**Math 392 Linear Algebra and Vector Analysis for Engineers** **Quiz 3**

1. Consider the following vector field on $\mathbb{R}^3$.

$$\vec{F}(x,y,z) = \langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2\rangle$$

(a) (4 points) Compute curl $\vec{F}$.

$$\text{curl}\,\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix} = \langle 6xyz^2 - 6xyz^2,\ 3y^2z^2 - 3y^2z^2,\ 2yz^3 - 2yz^3 \rangle = \boxed{\vec{0}}$$

(b) (4 points) Compute div $\vec{F}$.

$$\text{div}\,\vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle y^2z^3,\ 2xyz^3,\ 3xy^2z^2 \rangle$$

$$= 0 + 2xz^3 + 6xy^2z = \boxed{2xz^3 + 6xy^2z}$$

(c) (2 points) Is $\vec{F}$ conservative? Why or why not?

$\boxed{\text{YES.}}$ SINCE $\vec{F}$ IS DEFINED ON ALL OF $\mathbb{R}^3$ AND ITS COMPONENT FUNCTIONS HAVE CONTINUOUS PARTIAL DERIVATIVES, AND $\text{curl}\,\vec{F} = \vec{0}$, $\vec{F}$ IS CONSERVATIVE.

2. (8 points) Let $f$ be a scalar field and let $\vec{F}$ be a vector field. Circle the expressions below that are meaningful and cross out the expressions below that are not meaningful.

~~(a)~~ curl(div $\vec{F}$)

(b) grad(div $\vec{F}$)

(c) div(grad $f$)

~~(d)~~ div(div $\vec{F}$)

WE TAKE GRAD OF SCALARS & GET A VECTOR

WE TAKE CURL OF VECTORS & GET VECTORS

WE TAKE DIV OF VECTORS & GET A SCALAR

3. (8 points) Let $\vec{F} = \langle P(x,y,z), Q(x,y,z), R(x,y,z)\rangle$ be a vector field on $\mathbb{R}^3$ and suppose that $P, Q$ and $R$ have continuous second-order partial derivatives. Show that $\operatorname{div}\operatorname{curl}\vec{F} = 0$.

$$\text{CURL } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$\text{DIV } (\text{CURL } \vec{F}) = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \cdot \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$$

$$= R_{yx} - Q_{zx} + P_{zy} - R_{xy} + Q_{xz} - P_{yz} = 0$$

SINCE P, Q, R HAVE CONTINUOUS 2$^{nd}$ PARTIAL DERIVATIVES, MIXED PARTIALS ARE EQUAL.

4. (8 points) Find a parametric representation for the part of the cylinder $y^2 + z^2 = 16$ that lies between the planes $x = 0$ and $x = 5$.

( NOTE THAT YOU COULD THINK OF THIS AS A SURFACE OF REVOLUTION: THE GRAPH $y = f(x) = 4$ SPUN AROUND THE X-AXIS, $0 \le x \le 5$. )

$$\boxed{\vec{r}(x,\theta) = \langle x, 4\cos\theta, 4\sin\theta \rangle, \quad \underbrace{0 \le x \le 5, \quad 0 \le \theta \le 2\pi}}$$

THIS IS NOT <u>NECESSARY</u>.

WITHOUT IT, WE ASSUME $\theta \in \mathbb{R}$

AND THIS WORKS.

NOTATION:

$\vec{r}(u,v) = \langle u^2+1, v^3+1, u+v \rangle$

5. (8 points) Consider the following parametrized surface.

$$x = u^2 + 1, \quad y = v^3 + 1, \quad z = u + v.$$

Find an equation for the tangent plane to the surface at the point $(5, 2, 3)$.

FIRST WE FIND A NORMAL VECTOR TO THE SURFACE AT $(5,2,3)$, THAT IS A VECTOR PERPINDICULAR TO THE TANGENT PLANE.

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & 0 & 1 \\ 0 & 3v^2 & 1 \end{vmatrix} = \langle -3v^2, -2u, 6uv^2 \rangle$$

OBSERVE THAT $(5,2,3) = \vec{r}(2,1)$

SO OUR NORMAL VECTOR IS $\vec{r}_u \times \vec{r}_v (2,1) = \langle -3, -4, 12 \rangle$

$\therefore$ EQ OF TANGENT PLANE IS

$$-3x - 4y + 12z = D$$

SINCE $(5,2,3)$ IS IN THE PLANE,

$$-3(5) - 4(2) + 12(3) = -15 - 8 + 36 = 13 = D$$

$$\Rightarrow \quad \boxed{-3x - 4y + 12z = 13}$$

6. (8 points) Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

$S: \vec{r}(x,y) = \langle x, y, \underbrace{xy}_{\text{CALL THIS } g(x,y)} \rangle, \quad x^2 + y^2 \leq 1.$

(S IS A GRAPH)

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{g_x^2 + g_y^2 + 1} = \sqrt{x^2 + y^2 + 1}$$

$$\text{AREA} = \iint_S 1 \, dS = \iint_D |\vec{r}_x \times \vec{r}_y| \, dA = \iint_D \sqrt{x^2 + y^2 + 1} \, dA$$


LET $x = r\cos\theta \qquad 0 \leq r \leq 1$

$y = r\sin\theta \qquad 0 \leq \theta \leq 2\pi$

$x^2 + y^2 = r^2 \qquad dA = r\,dr\,d\theta$

$$= \int_0^{2\pi} \int_0^1 \sqrt{r^2 + 1} \; r\,dr\,d\theta$$

LET $u = r^2 + 1$

$\frac{1}{2} du = r\,dr$

$\int_{r=0}^{r=1} \sim \int_{u=1}^{u=2}$

$$= \int_0^{2\pi} \int_1^2 \frac{1}{2}\sqrt{u} \, du \, d\theta = \frac{2\pi}{3} u^{3/2} \Big|_1^2 = \boxed{\frac{2\pi}{3}\left(2\sqrt{2} - 1\right)}.$$