Name: * ANSWER KEY * | Due 5/15/2017
Math 392 Linear Algebra and Vector Analysis for Engineers | **Quiz 4**

1. Let A, B, C, D be the column matrices

$$A = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}, \quad B = \begin{bmatrix} -7 \\ 1 \\ -8 \end{bmatrix}, \quad C = \begin{bmatrix} -5 \\ -1 \\ 6 \end{bmatrix}, \quad D = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}.$$

For each of the following, state whether the matrix multiplication(s) can be performed. If not, state why not. If so, evaluate the expression.

(a) (4 points) $A^T A$

(b) (4 points) $AB^T + CD^T$

(c) (4 points) $AD - BC$

(a) $A^T A = \begin{bmatrix} 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = 4 + 1 \quad 4 = \boxed{9}$

(b) $AB^T + CD^T = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} -7 & 1 & -8 \end{bmatrix} + \begin{bmatrix} -5 \\ -1 \\ 6 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -14 & 2 & -16 \\ 7 & -1 & 8 \\ -14 & 2 & -16 \end{bmatrix} + \begin{bmatrix} -15 & -5 & 5 \\ -3 & -1 & 1 \\ 18 & 6 & -6 \end{bmatrix} = \boxed{\begin{bmatrix} -29 & -3 & -11 \\ 4 & -2 & 9 \\ 4 & 8 & -22 \end{bmatrix}}$

(c) A IS NOT CONFORMABLE WITH RESPECT TO D, AND
B " " " " C.

2. Consider the following system of liner equations.

$$\begin{array}{rcrcrcr} -2x & + & y & - & z & = & 4 \\ x & + & 2y & + & 3z & = & 13 \\ 3x & & & + & z & = & 1 \end{array}$$

(a) (2 points) Give the augmented matrix that corresponds to this system.

(b) (4 points) Give the reduced-row echelon form of the matrix from part (a).

(a) $\left[\begin{array}{ccc|c} -2 & 1 & -1 & 4 \\ 1 & 2 & 3 & 13 \\ 3 & 0 & 1 & 1 \end{array}\right]$

(b) SWAP R_1 & R_2 : $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ -2 & 1 & -1 & 4 \\ 3 & 0 & 1 & 1 \end{array}\right]$

$\begin{array}{r} \\ R_2 + 2R_1 : \\ R_3 - 3R_1 : \end{array} \left[\begin{array}{cccc} 1 & 2 & 3 & 13 \\ 0 & 5 & 5 & 30 \\ 0 & -6 & -8 & -38 \end{array}\right]$

$\frac{1}{5}R_2 : \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & -6 & -8 & -38 \end{array}\right]$

$R_3 + 6R_2 : \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & -2 & -2 \end{array}\right]$

$-\frac{1}{2}R_3 : \left[\begin{array}{ccc|c} 1 & 2 & 3 & 13 \\ 0 & 1 & 1 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$\begin{array}{r} R_1 - 3R_3 : \\ R_2 - R_3 : \\ \\ \end{array} \left[\begin{array}{cccc} 1 & 2 & 0 & 10 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$R_1 - 2R_2 : \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{array}\right]} \sim$

$x = 0$

$y = 5$

$z = 1$

3. (8 points) Describe the general solution to the following system as a linear combination of column vectors.

$$\begin{array}{rcrcrcrcr} w & + & x & - & y & + & z & = & -1 \\ 2w & + & 3x & - & 3y & + & 2z & = & -4 \\ -w & - & 2x & + & 2y & - & z & = & 3 \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -1 & 1 & -1 \\ 2 & 3 & -3 & 2 & -4 \\ -1 & -2 & 2 & -1 & 3 \end{array}\right] \quad \begin{array}{r} \\ R_2 - 2R_1: \\ R_3 + R_1: \end{array} \left[\begin{array}{ccccc} 1 & 1 & -1 & 1 & -1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & -1 & 1 & 0 & 2 \end{array}\right]$$

$$\begin{array}{r} R_1 - R_2: \\ \\ R_3 + R_2: \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\rightsquigarrow$ $w + z = 1 \longrightarrow w = 1 - z$

$x - y = -2 \longrightarrow x = -2 + y$

LET $y = s \in \mathbb{R}$ THEN $w = 1 - t$

$z = t \in \mathbb{R}$ $x = -2 + s$

$y = s$

$z = t$

i.e.

$$\boxed{\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}}$$

4. Let

$$A = \begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \quad \text{and} \quad B = \frac{1}{5}A^2A^T$$

(a) (8 points) Find det A.

(b) (4 points) Use the answer to part (a) to find det B.

(a)

$$\text{DET}\begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 3 & -4 & 1 \\ -1 & -2 & 0 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix} \quad \begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \end{matrix} \quad = \text{DET}\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 2 & 5 & 3 \end{bmatrix}$$

$$= \text{DET}\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 5 & 5 \end{bmatrix} \; (R_4 + 2R_2:) \quad = \quad \text{DET}\begin{bmatrix} 1 & 2 & -2 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 10 \end{bmatrix} \; (R_4 + \tfrac{5}{2}R_3:)$$

$$= (1)(-1)(-2)(10) = \boxed{20}$$

(b)

$$\text{DET}(B) = \left(\tfrac{1}{5}\right)^4 \text{DET}(A)^2 \cdot \text{DET}(A)$$

$$= \frac{20^3}{5^4} = \frac{8000}{625} = \boxed{\frac{64}{5} = 12.8}$$

5. (8 points) Find the values of λ for which the following matrix has determinant 0.

$$\begin{bmatrix} 1-\lambda & 0 \\ 3 & -3-\lambda \end{bmatrix}.$$

(Note: These are called the *eigenvalues* of the matrix $\begin{bmatrix} 1 & 0 \\ 3 & -3 \end{bmatrix}$.)

$(1-\lambda)(-3-\lambda) - (0)(3) = 0$

$(1-\lambda)(-3-\lambda) = 0$

$\boxed{\lambda = 1 \quad , \quad \lambda = -3}$

6. (8 points) Use Cramer's rule to solve the following system of equations.

$$\begin{aligned} 2x &+ 3y &= 2 \\ 4x &- 9y &= -1 \end{aligned}$$

$$\begin{bmatrix} 2 & 3 \\ 4 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} \boxed{\begin{matrix} 2 \\ -1 \end{matrix}} & \begin{matrix} 3 \\ -9 \end{matrix} \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -9 \end{vmatrix}} = \frac{(2)(-9) - (3)(-1)}{(2)(-9) - (3)(4)} = \frac{-18 + 3}{-18 - 12} = \frac{-15}{-30} = \frac{1}{2}$$

$$y = \frac{\begin{vmatrix} \begin{matrix} 2 \\ 4 \end{matrix} & \boxed{\begin{matrix} 2 \\ -1 \end{matrix}} \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 4 & -9 \end{vmatrix}} = \frac{(2)(-1) - (2)(4)}{(2)(-9) - (3)(4)} = \frac{-2 - 8}{-18 - 12} = \frac{-10}{-30} = \frac{1}{3}$$

$$\boxed{\begin{aligned} x &= \tfrac{1}{2} \\ y &= \tfrac{1}{3} \end{aligned}}$$

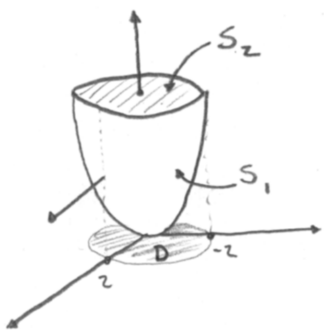
7. Let E be the solid bounded below by $z = x^2 + y^2$ and above by $z = 4$, and let S be the boundary surface of E with outward pointing normal. Let $\vec{F}$ be the vector field

$$\vec{F} = x\vec{i} + y\vec{j} + 1\vec{k}.$$

Calculate $\iint_S \vec{F} \cdot d\vec{S}$

(a) (8 points) directly as a surface integral, and

(b) (8 points) as a triple integral, by using the divergence theorem.



$$\iint_S \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

S_1 IS GRAPH $z = g(x,y) = x^2 + y^2$

$\vec{n}$ TO S_1 IS $\pm\langle -g_x, -g_y, 1\rangle$

$= \pm\langle -2x, -2y, 1\rangle$ CHOOSE $\ominus$ SO $\vec{n}$ POINTS OUTWARD

$$\Rightarrow \iint_{S_1} \vec{F}\cdot d\vec{S} = \iint_D \langle x, y, 1\rangle \cdot \langle 2x, 2y, -1\rangle\, dA = \iint_D (2x^2 + 2y^2 - 1)\, dA$$

$$= \int_0^{2\pi}\int_0^2 (2r^2 - 1)\, r\, dr\, d\theta = 2\pi\left(\tfrac{1}{2}r^4 - \tfrac{1}{2}r^2\right)\Big|_{r=0}^{r=2} = 12\pi$$

S_2 IS GRAPH $z = 4$ WITH $\vec{n} = \langle 0, 0, 1\rangle$

$$\Rightarrow \iint_{S_2} \vec{F}\cdot d\vec{S} = \iint_D \langle x, y, 1\rangle\cdot\langle 0,0,1\rangle\, dA = \iint_D dA = 4\pi$$

$\therefore \iint_S \vec{F}\cdot d\vec{S}$

$= 12\pi + 4\pi$

$= \boxed{16\pi}$

(b) $\operatorname{DIV} \vec{F} = 1 + 1 + 0 = 2$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{DIV}\vec{F}\, dV = 2\int_0^{2\pi}\int_0^2\int_{r^2}^4 r\, dz\, dr\, d\theta = 4\pi\int_0^2 (4 - r^2)\, r\, dr = 4\pi\left[2r^2 - \tfrac{1}{4}r^4\right]_0^2$$

$$= 4\pi\left[8 - 4\right] = \boxed{16\pi}$$