Name: *ANSWER KEY* | Due 5/25/2017
Math 392 Linear Algebra and Vector Analysis for Engineers | Quiz 5

1 Part 1: Answer question 1

1. (8 points) (a) Calculate the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 5 \\ 4 & 1 \end{bmatrix}.$$

(b) Use part (a) to solve the system of ordinary differential equations

$$\begin{aligned} y_1' &= 2y_1 + 5y_2 \\ y_2' &= 4y_1 + y_2 \end{aligned} \quad \text{with initial conditions} \quad \begin{aligned} y_1(0) &= 1 \\ y_2(0) &= 7 \end{aligned}.$$

(a) $(2-\lambda)(1-\lambda) - 20 = 0$

$\lambda^2 - 3\lambda - 18 = 0$

$(\lambda - 6)(\lambda + 3) = 0$

$\lambda = 6, -3$

$\boxed{\lambda_1 = 6:}$ $(A - \lambda I)\vec{x} = \begin{bmatrix} -4 & 5 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \Rightarrow -4x + 5y = 0$

$y = \frac{4}{5}x$

$\boxed{r\begin{bmatrix} 1 \\ \frac{4}{5} \end{bmatrix}, r \in \mathbb{R}}$

$\boxed{\lambda_2 = -3}$: $(A - \lambda I)\vec{x} = \begin{bmatrix} 5 & 5 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \vec{0} \Rightarrow 5x + 5y = 0$

$y = -x$

$\boxed{s\begin{bmatrix} 1 \\ -1 \end{bmatrix}, s \in \mathbb{R}}$

(b) $y(t) = re^{6t}\begin{bmatrix} 1 \\ 4/5 \end{bmatrix} + se^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$y_1(t) = re^{6t} + se^{-3t}$

$y_2(t) = \frac{4}{5}re^{6t} - se^{-3t}$

INIT. COND: $r + s = 1$

$\frac{4}{5}r - s = 7$

$\Rightarrow r = \frac{40}{9}, \ s = \frac{-31}{9}$

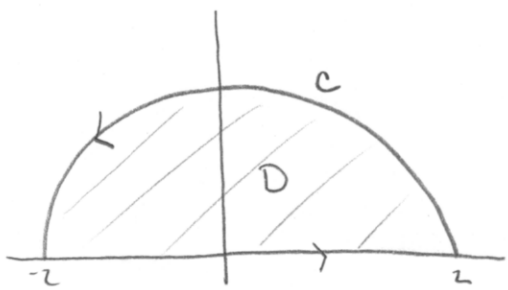
$$\boxed{\begin{aligned} y_1(t) &= \frac{40}{9}e^{6t} - \frac{31}{9}e^{-3t} \\ y_2(t) &= \frac{32}{9}e^{6t} + \frac{31}{9}e^{-3t} \end{aligned}}$$

2 Part 2: Answer 3 out of 4 of the following questions.

2. (8 points) A particle starts at the point $(-2, 0)$, moves along the x-axis to $(2, 0)$, and then along the semicircle $y = \sqrt{4 - x^2}$ back to the starting point. Find the work done on this particle by the force field

$$\vec{F}(x, y) = \langle x, x^3 + 3xy^2 \rangle .$$

(GREEN'S THM)
(EASY!)



BY GREEN'S THM:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$= \iint_D (3x^2 + 3y^2)\, dA = \int_0^{\pi} \int_0^{2} 3r^2 \cdot r \, dr \, d\theta$$

$$\frac{3\pi}{4} r^4 \Big|_0^2 = \boxed{12\pi}$$

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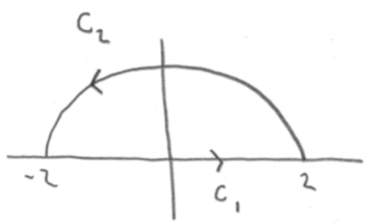
(DIRECTLY)
(HARD!)



$C_1 : \vec{r}(t) = \langle 4t - 2 , 0 \rangle , \quad 0 \le t \le 1$

$\vec{r}\,'(t) = \langle 4 , 0 \rangle$

$C_2 : \vec{r}(t) = \langle 2\cos t , 2\sin t \rangle , \quad 0 \le t \le \pi$

$\vec{r}\,'(t) = \langle -2\sin t , 2\cos t \rangle$

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \langle 4t-2 , (4t-2)^3 + 0 \rangle \cdot \langle 4 , 0 \rangle \, dt$$

$$= \int_0^1 16t - 8 \, dt = 8t^2 - 8t \Big|_0^1 = 0$$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^\pi \langle 2\cos t , (2\cos t)^3 + 3(2\cos t)(2\sin t)^2 \rangle \cdot \langle -2\sin t , 2\cos t \rangle \, dt$$

$$= \int_0^\pi -4\sin t \cos t + 16\cos^4 t + 48 \sin^2 t \cos^2 t \, dt$$

$$= \int_0^\pi \underbrace{-2\sin(2t)}_{0} + 16\cos^2 t \, \overbrace{(\cos^2 t + 3\sin^2 t)}^{1 + 2\sin^2 t} \, dt$$

$$= \int_0^\pi 16\cos^2 t + 32 \sin^2 t \cos^2 t \, dt = \int_0^\pi 8(1 + \cos 2t) + \overbrace{8\sin^2(2t)}^{4(1 - \cos 4t)} \, dt$$

$$= \int_0^\pi 12 + 8\cos(2t) - 4\cos(4t) \, dt = 12t + 4\sin(2t) - \sin(4t) \Big|_0^\pi$$

$$= \boxed{12\pi}$$

3. (8 points) Let

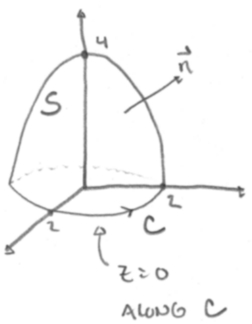
$$\vec{F} = x^2\vec{i} + (x+z)\vec{j} + yz\vec{k},$$

let S bet the surface of the paraboloid $z = 4 - x^2 - y^2$ that lies above the xy-plane, and let $\vec{n}$ be the upward pointing unit normal vector to S. Find the value of the surface integral

$$\iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,dS$$

(a) directly, and

(b) using Stoke's theorem.



(a) $S: \vec{r}(x,y) = \langle x, y, 4-x^2-y^2 \rangle$

$\vec{r}_x \times \vec{r}_y = \langle 2x, 2y, 1 \rangle$

$z = 4 - x^2 - y^2$

$$\text{CURL } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & x+z & yz \end{vmatrix} = \langle z-1, 0, 1 \rangle$$

$$\iint_S \text{curl } \vec{F}\cdot\vec{n}\,dS = \iint_D \langle 3-x^2-y^2, 0, 1\rangle \cdot \langle 2x, 2y, 1\rangle\, dA$$

$$= \iint_D \left(6x - 2x^3 - 2xy^2 + 1\right) dA = \int_0^{2\pi}\int_0^2 \left(\underbrace{6r\cos\theta}_{0} - \underbrace{2r^3\cos^3\theta}_{0} - \underbrace{2r^3\sin^2\theta\cos\theta}_{0} + 1\right) r\,dr\,d\theta$$

$$= \pi r^2 \Big|_0^2 = \boxed{4\pi}$$

(b) $C: \vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$

$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \langle 4\cos^2 t, 2\cos t, 0 \rangle \cdot \langle -2\sin t, 2\cos t \rangle\, dt$$

$$= \int_0^{2\pi} \underbrace{-8\sin t\cos^2 t}_{0} + 4\cos^2 t\, dt = 2\int_0^{2\pi} 1 + \underbrace{\cos(2t)}_{0}\, dt = \boxed{4\pi}$$

4. (8 points) Let

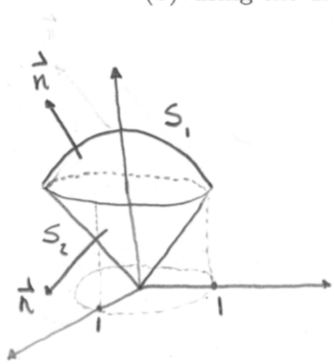
$$\vec{F} = x\vec{i} + y\vec{j} + z\vec{k},$$

let S be the surface of the region between the graphs of $z = \sqrt{2 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$, and let $\vec{n}$ be the outward pointing unit normal vector to S (Note that $\vec{n}$ points upward from the top surface and downward from the bottom surface). Find the value of the surface integral

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

(a) directly, and

(b) using the divergence theorem.



$S_1: \vec{r}(x,y) = \langle x, y, \sqrt{2 - x^2 - y^2} \rangle$

$\vec{r}_x \times \vec{r}_y = \left\langle \dfrac{x}{\sqrt{2 - x^2 - y^2}}, \dfrac{y}{\sqrt{2 - x^2 - y^2}}, 1 \right\rangle$

$$\iint_{S_1} \vec{F} \cdot \vec{n}\, dS = \iint_D \langle x, y, \sqrt{2 - x^2 - y^2} \rangle \cdot \left\langle \frac{x}{\sqrt{2 - x^2 - y^2}}, \frac{y}{\sqrt{2 - x^2 - y^2}}, 1 \right\rangle dA$$

$$= \iint_D \frac{x^2 + y^2}{\sqrt{2 - x^2 - y^2}} + \sqrt{2 - x^2 - y^2}\, dA = \iint_D \frac{2}{\sqrt{2 - x^2 - y^2}}\, dA = \int_0^{2\pi}\int_0^1 \frac{2r}{\sqrt{2 - r^2}}\, dr\, d\theta$$

$$= -2\pi \int_1^2 \frac{1}{\sqrt{u}}\, du = 4\pi \sqrt{u}\,\Big|_1^2 = 4\pi(\sqrt{2} - 1)$$

$S_2: \vec{r}(x,y) = \langle x, y, \sqrt{x^2 + y^2} \rangle$

$-\vec{r}_x \times \vec{r}_y = \left\langle \dfrac{x}{\sqrt{x^2 + y^2}}, \dfrac{y}{\sqrt{x^2 + y^2}}, -1 \right\rangle$

POINT OUTWARD

$$\iint_{S_2} \vec{F} \cdot \vec{n}\, dS = \iint_D \langle x, y, \sqrt{x^2 + y^2} \rangle \cdot \left\langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}}, -1 \right\rangle dA$$

$$= \iint_D \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} - \sqrt{x^2 + y^2}\, dA = 0$$

$$\therefore \iint_S \vec{F} \cdot \vec{n}\, dS = \boxed{4\pi(\sqrt{2} - 1)}$$

(b) $\text{DIV } \vec{F} = 1 + 1 + 1 = 3$

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iiint_E 3\, dV = 3 \int_0^{2\pi} \int_0^{1} \int_r^{\sqrt{2-r^2}} r\, dz\, dr\, d\theta$$

$$= 6\pi \int_0^1 \left(\sqrt{2-r^2} - r \right) r\, dr = 6\pi \left[\int_0^1 r\sqrt{2-r^2}\, dr - \int_0^1 r^2\, dr \right]$$

$$= 6\pi \left[\frac{1}{2} \int_1^2 \sqrt{u}\, du - \frac{1}{3} r^3 \Big|_0^1 \right]$$

$$= 6\pi \left[\frac{1}{3} u^{3/2} \Big|_1^2 - \frac{1}{3} \right] = 2\pi \left[2^{3/2} - 2 \right] = \boxed{4\pi \left(\sqrt{2} - 1 \right)}$$

OR, USING SPHERICAL COORD.,

$$3 \int_0^{2\pi} \int_0^{\pi/4} \int_0^{\sqrt{2}} \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \cdot 2^{3/2} \int_0^{\pi/4} \sin\phi\, d\phi$$

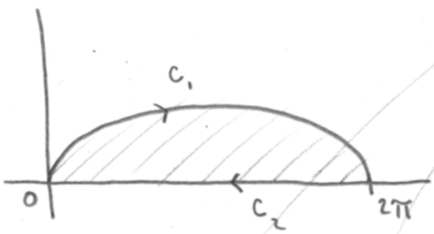
$$= -2^{5/2} \pi \cos\phi \Big|_0^{\pi/4} = -4\sqrt{2}\pi \left(\frac{\sqrt{2}}{2} - 1 \right)$$

$$= \boxed{4\pi \left(\sqrt{2} - 1 \right)}$$

5. (8 points) Use Green's theorem to find the area under one arch of the cycloid given by the parametric equations

$$C_1 : \begin{array}{rcl} x(t) & = & t - \sin t \\ y(t) & = & 1 - \cos t \end{array} \quad ; \quad 0 \leq t \leq 2\pi.$$

That is, find the area of the region enclosed by $C = C_1 + C_2$, where C_2 is the line segment from $(2\pi, 0)$ to $(0,0)$. Note that C is oriented *clockwise.*



$x'(t) = 1 - \cos t$

$y'(t) = \sin t$

$$\text{AREA} = -\int_{C_1} x\,dy - \int_{C_2} x\,\overbrace{dy}^{0} = -\int_0^{2\pi} (t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi} t\sin t\,dt + \underbrace{\int_0^{2\pi} \sin^2 t\,dt}_{\pi}$$

$u = t \qquad v = \cos t$

$du = dt \qquad dv = -\sin t\,dt$

$$= t\cos t\Big|_0^{2\pi} - \underbrace{\int_0^{2\pi} \cos t\,dt}_{0} + \pi = 2\pi - 0 + \pi = \boxed{3\pi}$$