Fractions

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Introduction

Mathematics begins with the *group* of integers, $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3, \ldots\}$, paired with the operation of addition, $+$. This is a closed system; that is, given any two integers a and b, their sum, $a + b$, is also an integer. In the group of integers, 0 is distinguished as the only integer such that $a + 0 = a$ for all integers a. For this reason, 0 is referred to as the additive identity. Furthermore, for every integer a, there is an integer, denoted $-a$, such that $a + (-a) = 0$. This says the following.

> For every integer a there exists another integer $-a$, the additive inverse of a, such that their sum equals the additive identity, 0.

Note that subtraction is actually a particular kind of addition; that is, $a - b := a + (-b)$. The point of all this is that the integers are sufficient for solving integer equations involving addition.

$$a + x = b$$

$$x = b - a$$

Repeated addition leads to the idea of introducing a new operation, multiplication. Given any two integers a and b, their product $a \cdot b$, or simply ab, is also an integer. And here there is another distinguished integer, 1, which is the unique integer such that $a \cdot 1 = a$ for all integers a. For this reason, 1 is referred to as the multiplicative identity.

Now we ask the following question.

> Given any non-zero integer a, is there another integer b such that their product is the multiplicative identity, 1?

With two exceptions (what are they), the answer is generally no. So the integers paired with multiplication is not a closed system (i.e. *group*). With some exceptions (what are they?), the integers are not sufficient for solving integer equations involving multiplication.

$$ax = b$$

$$x = ?$$

$$\vdots$$

$$x = \frac{b}{a}$$

And so the idea of fractions is born!

From now on, we leave the group of integers behind and work instead with the larger *field* of rational numbers $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}\}$. Note that the set of integers is contained inside the set of rational numbers, for $a = a/1$ for every integer a.

Multiplying Fractions

Let a, b, c, and d be integers, and $b, d \neq 0$. Then

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Note also that (and explain to yourself why) the following are equivalent.

$$a \cdot \frac{b}{c} = \frac{ab}{c} = \frac{a}{c} \cdot b$$

Practice: p. 49-50[1]

Equivalent Fractions

By definition (and also by the formula given in the last section),

$$\frac{a}{a} = 1 \text{ for all integers } a \neq 0.$$

Thus, the number 1 has infinitely many representatives within the set of rational numbers. Furthermore,

$$\frac{p}{q} = \frac{ap}{aq} \text{ for all integers } a, p, q \text{ with } a, q \neq 0.$$

This says that rational numbers remain unchanged when both numerators and denominators are multiplied by the same quantity. Reading the above equation from right to left, we see how fractions can be "simplified" or "reduced".

Practice: p. 53-54

[1] Page numbers refer to "Developmental Mathematics", ed. 4, by Johnston, Willis, and Hughes.

Prime Factorization

We begin with a few definitions from number theory. Let a be a positive integer.

> The *multiples* of a consist of all integers which can be written as na for some non-negative integer n.
>
> The *factors*, i.e *divisors*, of a consist of all integers m and n such that $a = mn$.

For example, 12 is a multiple of 4, and 4 is a factor of 12.

Note that every positive integer has infinitely many multiples, but only finitely many factors, and the factors come in pairs (So does every positive integer have an even number of factors?).

Remark. There are "tricks" for knowing when a given number is divisible by 2, 3, 4, 5, 6, 8, 9, and 10.

> *Prime* numbers (or just *primes*) are positive integers p whose only divisor are 1 and p.

The Fundamental theorem of arithmetic states that every positive integer is either prime or can be written as a unique product of primes, called its *prime factorization*. Note that when prime factors are repeated, we use exponent notation, e.g. $40 = 2 \cdot 2 \cdot 2 \cdot 5 = 2^3 \cdot 5$.

Practice: p. 56

Greatest Common Factor

Let a and b be two positive integers.

> The *greatest common factor* (i.e. GCF) of a and b, denoted $\gcd(a, b)$, is the greatest number that is both a factor of a and a factor of b.

Remark. When both a and b have been factored into primes, $\text{gcf}(a, b)$ is simply the product of all *common* prime factors raised to the *lowest* exponent.

Reducing Fractions

> A fraction p/q is called *reduced*, or *simplified*, when $\text{gcf}(p, q) = 1$.

Thus, to reduce a given fraction a/b, we simply divide both the numerator and denominator by $\text{gcf}(a, b)$.

$$\frac{a}{b} \text{ reduced} = \frac{a \div \text{gcf}(a, b)}{b \div \text{gcf}(a, b)}$$

Practice: p. 59

Practice: p. 60 (cross canceling)

Dividing Fractions

Let a, b, c, and d be integers, and let $b, c, d \neq 0$. Then

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

This is sometimes referred to as "switch and flip".

Practice: p. 74 #1-25

Adding & Subtracting Like Fractions

Let a, b, c be integers, with $c \neq 0$. Then

$$\frac{a}{c} \pm \frac{b}{c} = \frac{a \pm b}{c}.$$

Practice: p. 62

Least Common Multiple & Lowest Common Denominator

Let a and b be two positive integers.

> The *least common multiple* (LCM) of a and b, denoted $\text{lcm}(a, b)$, is the smallest number which is both a multiple of a and a multiple of b.

Remark. When both a and b have been factored into primes, $\text{lcm}(a, b)$ is simply the product of *all* prime factors raised to the *highest* exponent.

Practice: p. 64

> Given two or more fractions, the *lowest common denominator* (LCD) is the LCM of the denominators. It is the smallest denominator that can be used to express all of the fractions.

Remember that when generating equivalent fractions, we need to multiply both denominator *and* numerator by the same non-zero number.

Practice: p. 67

Complex Fractions

A *simple fraction* is a fraction that has only one fraction line. A *complex fraction* is a fraction that has more than one fraction line. That is, a complex fraction is a fraction whose numerator or denominator contains a fraction.

There are two methods of turning a simple fraction into a complex fraction.

1. Turn the numerator and denominator into simple fractions and then perform the division.

2. Multiply both the numerator and denominator by the LCM of all "inside" denominators.

Practice: p. 76

Powers & Roots of Fractions

Let a and b be integers, $b \neq 0$, and let n be a non-negative integer. Then

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

and

$$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Practice: p. 77-78

Improper Fractions & Mixed Numbers

Let a and b both be positive integers.

A *proper fraction* is a fraction $\dfrac{a}{b}$ such that $a < b$.

An *improper fraction* is a fraction $\dfrac{a}{b}$ such that $a \geqslant b$.

In the latter case, we can use the *division algorithm* to find integers q (quotient) and r (remainder), with $0 \leqslant r < b$, such that

$$a = qb + r,$$

i.e.,

$$\frac{a}{b} = q + \frac{b}{r}.$$

Remark. Generally, the + sign is omitted from the fraction on the right above, and this sum of an integer and a proper fraction is called a *mixed number*. For example, $2\frac{3}{8}$ is mixed number that is equivalent to the improper fraction $\frac{19}{8}$.

Practice: p. 51 (improper fraction to mixed number)
Practice: p. 52 (mixed number to improper fraction)
Practice: p. 69 (adding mixed numbers, two ways!)
Practice: p. 71 (subtracting mixed numbers, two ways!)
Practice: p. 74 #26-35 (diving mixed numbers)

Combined Operation

When a calculation involves multiple operations, the operations are performed in the following order:

1. Parenthesis
2. Exponents/roots
3. Multiplication/division, left to right
4. Addition/subtraction, left to right

Remark. Of course, the *distributive property* allows us to alter this order slightly.

Practice: p. 80

Comparing Fractions

When comparing two fractions, the following fact can be used.

$$\frac{a}{b} \geqslant \frac{c}{d} \text{ if and only if } ad \geqslant bc.$$

This is sometimes referred to as "cross multiplication".

When comparing three or more fractions, it is usually best to instead compare equivalent fractions with identical denominators (ideally, the LCD).

Practice p. 81