§3.2 COMPOUND INTEREST

2 SIBLINGS EACH INVEST $1000 INTO ACCOUNTS THAT EARN 10% INTEREST FOR 5 YEARS. ONE EARNS SIMPLE INTEREST, ONE EARNS ANNUAL

COMPOUND INTEREST - INTEREST EARNED ON PRINCIPAL AND ANY PREVIOUSLY EARNED INTEREST.

YEARS LATER	SIMPLE INTEREST		ANNUAL COMPOUND INTEREST	
0	A_0 P	$= 1000$	$A_0 = P$	$= 1000$
1	$A_1 = A_0 + Pr = P(1+r)$	$= 1100$	$A_1 = A_0 + A_0 r = P(1+r)$	$= 1100$
2	$A_2 = A_1 + Pr = P(1+2r)$	$= 1200$	$A_2 = A_1 + A_1 r = A_1(1+r) = P(1+r)^2$	$= 1210$
3	$A_3 = A_2 + Pr = P(1+3r)$	$= 1300$	$A_3 = A_2 + A_2 r = A_2(1+r) = P(1+r)^3$	$= 1331$
4	$A_4 = A_3 + Pr = P(1+4r)$	$= 1400$	$A_4 = A_3 + A_3 r = A_3(1+r) = P(1+r)^4$	$= 1464.10$
5	$A_5 = A_4 + Pr = P(1+5r)$	$= 1500$	$A_5 = A_4 + A_4 r = A_4(1+r) = P(1+r)^5$	$= 1610.51$

$$A = P(1+rt) \qquad\qquad A = P(1+r)^t$$

$$(I = A - P)$$

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ex. SUPPOSE $800 IS INVESTED INTO ACCOUNT EARNING ANNUAL COMPOUND INTEREST.
10 YRS LATER IT IS WORTH $1,278.51.
WHAT IS THE INTEREST RATE?

INTEREST IS COMPOUNDED WHEN IT BECOMES PART OF THE AMOUNT UPON WHICH INTEREST IS CALCULATED.

IF COMPOUNDED k TIMES PER YEAR, THEN AMOUNT EARNS INTEREST RATE $\frac{r}{k}$ EACH COMPOUND PERIOD.

$$A = P\left(1+\frac{r}{k}\right)^{kt}$$

P = PRINCIPAL
r = ANNUAL INTEREST RATE
k = # COMPOUND PERIODS EACH YEAR
t = # YEARS

$$A = P(1+i)^n$$

$i = \frac{r}{k}$, $n = kt$ = # COMPOUND PERIODS

ex. $1000 @ 10% COMPOUNDED ANNUALLY FOR 5 YEARS → 1610.51
$1000 @ 10% COMPOUNDED QUARTERLY FOR 5 YEARS → 1638.62

ex. IF $P GROWS TO $3000 IN 18 MONTHS WHILE EARNING 6% INTEREST COMPOUNDED MONTHLY, FIND P.

PRESENT VALUE $P = A(1+i)^{-n}$

LOGARITHMS

$$a^x = y \iff \log_a y = x$$

ex. $2^3 = 8 \iff \log_2 8 = 3$

ex. $25^{1/2} = 5 \iff \log_{25} 5 = \frac{1}{2}$

NOTE: CHANGE OF BASE FORMULA

$$\log_a y = \frac{\log y}{\log a} = \frac{\ln y}{\ln a}$$

SO, $A = P(1+i)^n \iff (1+i)^n = \frac{A}{P}$

$$\iff \log_{(1+i)}\left(\frac{A}{P}\right) = n \iff n = \frac{\ln(A/P)}{\ln(1+i)}$$

ex. HOW LONG WILL IT TAKE FOR AN INVESTMENT TO DOUBLE @ 3% INTEREST COMPOUNDED SEMIANNUALLY?

3. At what rate will money double in 25 years, if interest is compounded annually? (*Ans.* 2.8 per cent.)

4. The same, if interest is compounded semi-annually?

5. What interest rate, compounded quarterly, will have to be obtained if $5000 must amount to $6500 in 5 years? (*Ans.* 5.3 per cent.)

6. In how many years will money double at 4 per cent compounded annually? (*Ans.* 17.68.)

7. How long will it take $1000 to amount to $1250 at 3 per cent compounded semi-annually?

8. Solve the equation $7^x = 11$.

9. The Dutch purchased Manhattan Island from the Indians in 1626 for $24. To what sum would this have amounted in 1936 if interest had been at 6 per cent?

10. What sum should be set aside on a boy's first birthday in order to provide an education fund of $4000 when he is 18, if 4 per cent interest, compounded quarterly, was obtained throughout the period?

11. A factory management must plan to replace certain machinery every 15 years. If the machinery will always cost $5000 and interest is always at 3 per cent, what

sum should be set aside when the machinery is first put in to provide for its perpetual replacement? (The sum will have to produce itself plus $5000 every 15 years.) (*Ans.* $8961.)

12. A $1500 automobile loses each year 30 per cent of its value at the beginning of that year. What is it worth after 4 years? [This is a problem in *depreciation;* the rate in (12) is negative.] (*Ans.* $360.)

13. An investment depreciates, losing each year 6 per cent of its value at the beginning of the year. In how many years will it have shrunk to half its original value?

Effective Rate (APY)

If you invest $100 at 9%, compounded monthly, then your balance at the end of one year is

$$A = P(1+i)^n = 100\left(1+\frac{.09}{12}\right)^{12} = \$109.38.$$

You have earned $9.38 in interest, which is 9.38% of your original $100. In other words, $100 invested at 9.38% compounded *annually* will produce the same amount of interest (namely, $100 * .0938 = $9.38) as does 9% compounded monthly. In this situation, 9% is called the **nominal** or **stated rate,** while 9.38% is called the **effective rate** or **annual percentage yield** **(APY).**

In the discussion that follows, the nominal rate is denoted r and the APY (effective rate) is denoted r_E.

$$P(1+r_E)^1 = P\left(1+\frac{r}{K}\right)^K$$

$$1+r_E = \left(1+\frac{r}{K}\right)^K$$

$$r_E = \left(1+\frac{r}{K}\right)^K - 1$$

Effective Rate (r_E) or Annual Percentage Yield (APY)

The *APY* r_E is the annual compounding rate needed to produce the same amount of interest in one year, as the nominal rate does with more frequent compounding.

ex. FIND APY FOR A PRINCIPAL THAT EARNS 100% COMPOUNDED

- ANNUALLY 100%
- SEMIANNUALLY 125%
- QUARTERLY 144.14%
- MONTHLY 161.30%
- DAILY 171.46%