

§8.4 Bayes' Formula

1 The Law of Total Probability

Example 1. The following table gives the proportion of NYC residents that live in each borough, and the proportion of borough residents that were born outside the United States. What proportion of NYC residents were born outside the United States?

	M	BK	Q	BX	SI
Proportion of NYC residents	.19	.31	.27	.17	.06
Proportion born outside US	.29	.38	.49	.32	.21

2 Bayes' Formula

Example 2. Using the data from the previous question, what is the probability that a randomly selected NYC resident that was born outside of the United States lives in Queens?

THEOREM 1 Bayes' Formula

Let U_1, U_2, \dots, U_n be n mutually exclusive events whose union is the sample space S . Let E be an arbitrary event in S such that $P(E) \neq 0$. Then,

$$\begin{aligned} P(U_1|E) &= \frac{P(U_1 \cap E)}{P(E)} = \frac{P(U_1 \cap E)}{P(U_1 \cap E) + P(U_2 \cap E) + \dots + P(U_n \cap E)} \\ &= \frac{P(E|U_1)P(U_1)}{P(E|U_1)P(U_1) + P(E|U_2)P(U_2) + \dots + P(E|U_n)P(U_n)} \end{aligned}$$

Similar results hold for U_2, U_3, \dots, U_n .

Example 3. Suppose a test for a disease is 99% accurate.

- The probability that a person that has the disease tests positive is .99.
- The probability that a person that does not has the disease tests negative is .99.

Suppose 1% of the general population has the disease. What is the probability that a person who tests positive for the disease has the disease?

Example 4. The following table summarizes data about a population of student commuters on the first day of school.

	Walk	Bike	Train/Bus	Car
Proportion of population	.06	.09	.72	.13
Proportion that arrive late	.02	.04	.18	.08

1. Overall, what is the probability that a student commuter from this population arrives late on the first day of school?

2. Given that a student commuter is late on the first day of school, what is the probability that they took the train/bus?

3 More examples

Example 5. City crime records show that 15% of all crimes are violent and 85% are non-violent, involving theft, forgery, and so on. Additionally, 90% of violent crimes are reported versus 60% of nonviolent crimes.

1. What is the overall reporting rate for crimes in the city?
2. If a crime in progress is reported to the police, what is the probability that the crime is violent?

Example 6. In answering a question on a multiple-choice test, a student either knows the answer or guesses. Let p be the probability that the student knows the answer and $1 - p$ be the probability that the student guesses. Assume that a student who guesses at the answer will be correct with probability $1/m$, where m is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he or she answered it correctly?

Example 7. A laboratory blood test is 95% effective in detecting a certain disease when it is, in fact, present. However, the test also yields a “false positive” result for 1% of the healthy persons tested. (That is, if a healthy person is tested, then, with probability .01, the test result will imply that he or she has the disease.) If .5% of the population actually has the disease, what is the probability that a person has the disease given that the test result is positive?

Example 8. At a certain stage of a criminal investigation, the inspector in charge is 60% convinced of the guilt of a certain suspect. Suppose, however, that a new piece of evidence which shows that the criminal has a certain characteristic (such as left-handedness, baldness, or brown hair) is uncovered. If 20% of the population possesses this characteristic, how certain of the guilt of the suspect should the inspector now be if it turns out that the suspect has the characteristic?

Example 9. An ectopic pregnancy is twice as likely to develop when the pregnant woman is a smoker as it is when she is a nonsmoker. If 32 percent of women of childbearing age are smokers, what percentage of women having ectopic pregnancies are smokers?

Requires the Law of Total Probability

Example 10. To test the efficacy of an experimental drug at curing a particular disease, volunteer patients with the disease were first split into 3 categories according to the severity of their disease and then given the experimental drug.

- 70% of patients had a mild case of the disease, and 95% of these patients recovered.

- 25% of patients had a moderate case of the disease, and 35% of these patients recovered.
- 5% of patients had a severe case of the disease, and 15% of these patients recovered.

1. What percent of all patients recovered from the disease?
2. Given that a patient recovered, what is the probability that they had a severe case of the disease?

Example 11. A bin contains 3 different types of disposable flashlights. The probability that a type 1 flashlight will give over 100 hours of use is .7, with the corresponding probabilities for type 2 and type 3 flashlights being .4 and .3, respectively. Suppose that 20% of the flashlights in the bin are type 1, 30% are type 2, and 50% are type 3.

1. What is the probability that a randomly chosen flashlight will give more than 100 hours of use?
2. Given that a flashlight lasted over 100 hours, what is the conditional probability that it was a type j flashlight, $j = 1, 2, 3$?

Example 12. A total of 46% of the voters in a certain city classify themselves as Independents, whereas 30% classify themselves as Liberals and 24 percent say that they are Conservatives. In a recent local election, 35% of the Independents, 62% of the Liberals, and 58% of the Conservatives voted. A voter is chosen at random. Given that this person voted in the local election, what is the probability that he or she is

1. an Independent?
2. a Liberal?
3. a Conservative?
4. What fraction of voters participated in the local election?

Bonus questions

Example 13. A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability p or moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent.

1. What is the probability that after 2 days the stock will be at its original price?
2. What is the probability that after 3 days the stock's price will have increased by 1 unit?
3. Given that after 3 days the stock's price has increased by 1 unit, what is the probability that it went up on the first day?

Example 14. A true-false question is to be posed to a husband-and-wife team on a quiz show. Both the husband and the wife will independently give the correct answer with probability p . Which of the following is a better strategy for the couple?

1. Choose one of them and let that person answer the question.
2. Have them both consider the question, and then either give the common answer if they agree or, if they disagree, flip a coin to determine which answer to give.