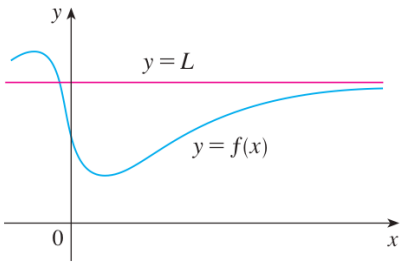
§3.4 Limits at Infinity & Horizontal Asymptotes

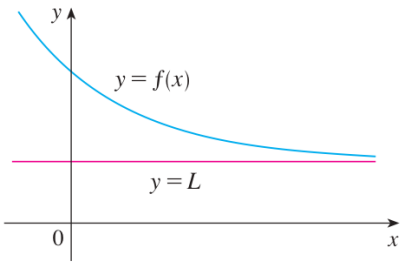
1 Intuitive Definition of a Limit at Infinity Let f be a function defined on some interval (a, ∞). Then

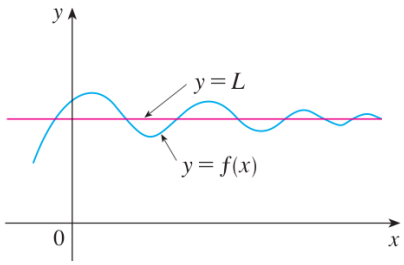
$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large.

INPUT GETS LARGER & LARGER, OUTPUT GETS CLOSER & CLOSER TO L





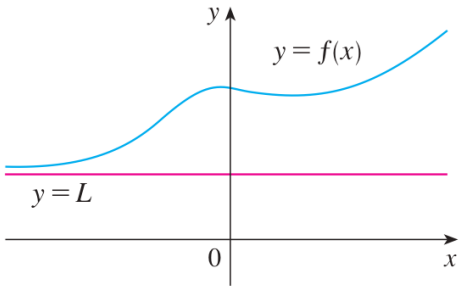


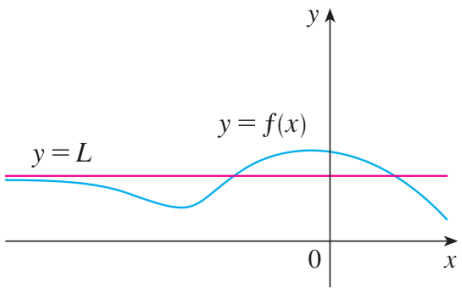
$$y = \frac{1}{1+x^2} \sin(x)$$

2 Definition Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x \to -\infty} f(x) = L$$

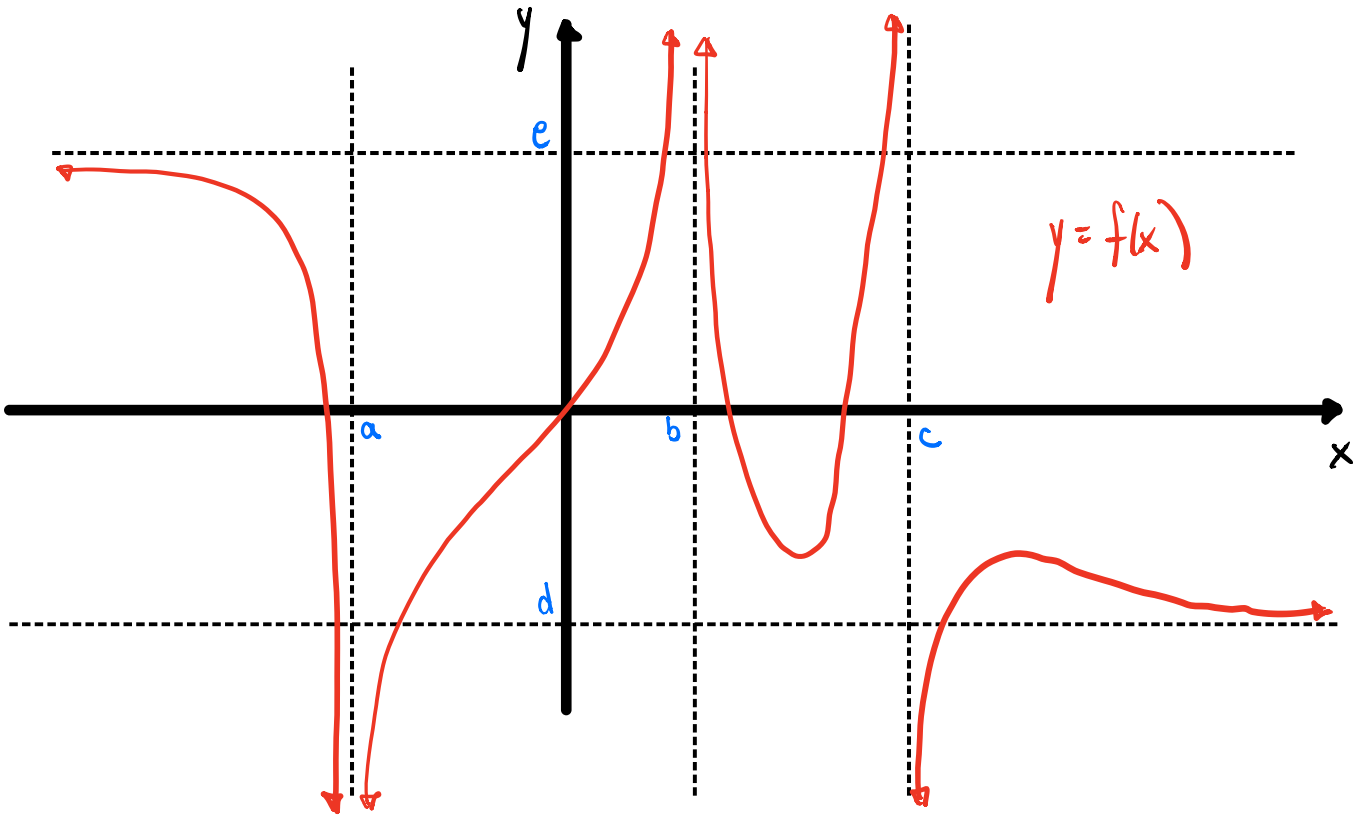
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x to be sufficiently large negative.





3 Definition The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = L$$



HORIZONTAL ASYMPTOTES: $\lim_{x \to \infty} f(x) = d \rightarrow y = d$ is H.A.

$\lim_{x \to -\infty} f(x) = e \rightarrow y = e$ is H.A.

VERTICAL ASYMPTOTES: $\lim_{x \to a} f(x) = -\infty \rightarrow x = a$ is V.A.

$\lim_{x \to b} f(x) = \infty \rightarrow x = b$ is V.A.

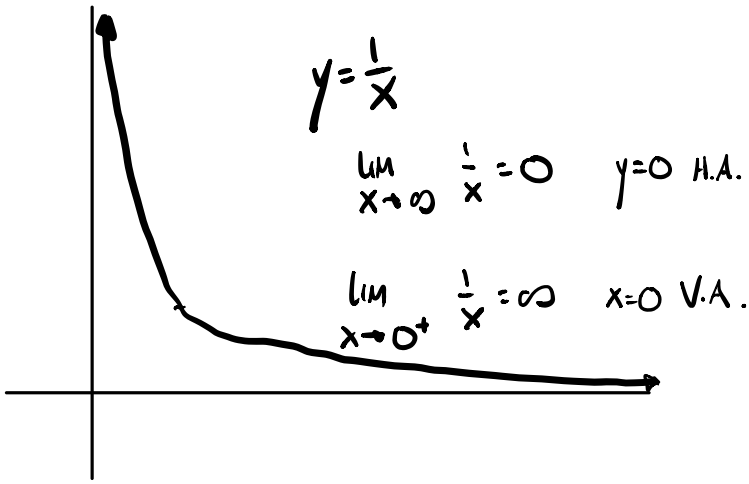
$\lim_{x \to c^-} f(x) = \infty \rightarrow$

$\lim_{x \to c^+} f(x) = -\infty \rightarrow$

$\Big\}$ $x = c$ is V.A.

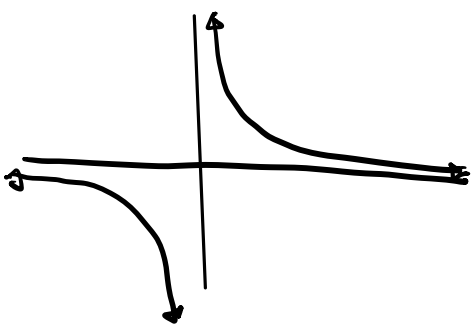
ex. FIND $\lim_{x\to\infty} \frac{1}{x}$, $\lim_{x\to-\infty} \frac{1}{x}$.

x	$f(x) = \frac{1}{x}$
10	.1
100	.01
1000	.001
10000	.0001
$\downarrow$	$\downarrow$
∞	0



x	$f(x) = \frac{1}{x}$
.1	10
.01	100
.001	1000
.0001	10000

$f(-x) = -f(x)$



4 Theorem If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0$$

If $r > 0$ is a rational number such that x^r is defined for all x, then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0$$

$\lim_{x\to\infty} x^r = \infty$, $r > 0$

$$\lim_{x\to\infty} \frac{1}{x^{1/n}} = 0 \quad , n>0 \qquad \lim_{x\to-\infty} \frac{1}{x^{1/n}} = \text{D.N.E.}$$

ex. FIND $\lim_{x\to\infty} \dfrac{3x^6 - 4x^3 + 2x}{4x^6 + 2x^4 - 3x^2 + 1}$

RATIONAL FUNC. (RATIO OF POLYNOMIALS)

DEG of Top = DEG Bottom

$$= \lim_{x\to\infty} \frac{\cancel{x^6}\left(3 - \frac{4}{x^3} + \frac{2}{x^5}\right)}{\cancel{x^6}\left(4 + \frac{2}{x^2} - \frac{3}{x^4} + \frac{1}{x^6}\right)}$$

FORCE FACTORING

$$= \frac{\lim_{x\to\infty} 3 - \frac{4}{x^3} + \frac{2}{x^5}}{\lim_{x\to\infty} 4 + \frac{2}{x^2} - \frac{3}{x^4} + \frac{1}{x^6}} = \frac{3 - 0 + 0}{4 + 0 - 0 + 0}$$

$$= \frac{3}{4}$$

ex. FIND $\lim_{x\to\infty} \sqrt{25x^2 + 8x} - 5x$ $\quad (\infty - \infty$ UNDEFINED$)$

$$\lim_{x\to\infty} \frac{\left(\sqrt{25x^2+8x} - 5x\right)\left(\sqrt{25x^2+8x} + 5x\right)}{\left(\sqrt{25x^2+8x} + 5x\right)}$$

$$= \lim_{x\to\infty} \frac{\sqrt{25x^2+8x}^2 - (5x)^2}{\left(\sqrt{25x^2+8x} + 5x\right)}$$

$$\left(\sqrt{25x^2+8x}+5x\right)$$

$$= \lim_{x\to\infty} \frac{25x^2+8x-25x^2}{\left(\sqrt{25x^2+8x}+5x\right)}$$

Note: $\sqrt{25x^2+8x} = \sqrt{25x^2\left(1+\frac{8}{25x}\right)}$

$$= \sqrt{25x^2}\sqrt{1+\frac{8}{25x}}$$

$$= 5|x|\sqrt{1+\frac{8}{25x}}, \quad x>0$$

$$= 5x\sqrt{1+\frac{8}{25x}}$$

$$= \lim_{x\to\infty} \frac{8x}{5x\sqrt{1+\frac{8}{25x}}+5x}$$

$$= \lim_{x\to\infty} \frac{8x}{5x\left(\sqrt{1+\frac{8}{25x}}+1\right)} = \frac{8}{5}\left[\frac{1}{\sqrt{1+\frac{8}{25}\underbrace{\lim_{x\to\infty}\frac{1}{x}}_{0}}+1}\right]$$

$$= \frac{8}{5}\left[\frac{1}{\sqrt{1+0}+1}\right] = \frac{8}{5}\cdot\frac{1}{2} = \frac{4}{5} = .8$$

Infinite Limits at Infinity

The notation

$$\lim_{x\to\infty} f(x) = \infty$$

is used to indicate that the values of $f(x)$ become large as x becomes large. Similar meanings are attached to the following symbols:

$$\lim_{x\to-\infty} f(x) = \infty \qquad \lim_{x\to\infty} f(x) = -\infty \qquad \lim_{x\to-\infty} f(x) = -\infty$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

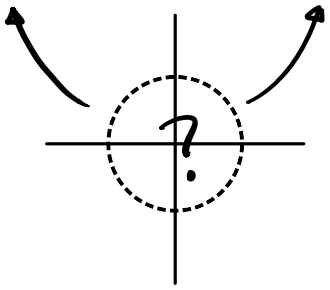
be an n^{th} degree polynomial with lead coefficient a_n.

Then

When $|x|$ is large $f(x) \approx a_n x^n$

n even

$a_n > 0$



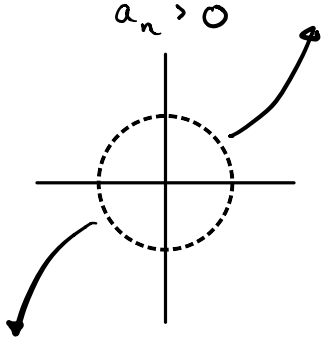
e.g. $f(x) = x^2$

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = \infty$

n odd

$a_n > 0$



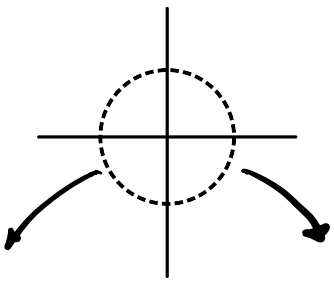
e.g. $f(x) = x^3$

$\lim_{x \to \infty} f(x) = \infty$

$\lim_{x \to -\infty} f(x) = -\infty$

n even

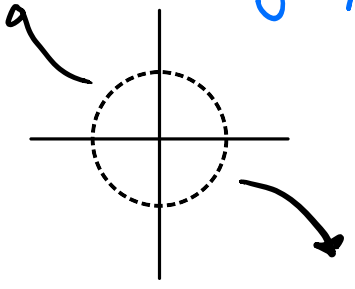
$a_n < 0$



e.g. $y = -x^2$

$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = -\infty$



e.g. $y = -x^3$

$\lim_{x \to \infty} f(x) = -\infty$

$\lim_{x \to -\infty} f(x) = \infty$

ex. SKETCH $y = x^2(x+3)^3(x-4)^4(x-10)^5$, FOCUS ON INTERCEPTS & LIMITS AT INFINITY $\left(\lim_{x \to \pm\infty}\right)$

WHEN $|x|$ IS LARGE

$$y \approx x^2(x^3)(x^4)(x^5) = x^{14}$$

so $\lim_{x \to \infty} y = \lim_{x \to -\infty} y = \infty$

$y = x^{(2)}(x+3)^{3}(x-4)^{(4)}(x-10)^{5}$ → x-INTERCEPTS: $-3, 0, 4, 10$

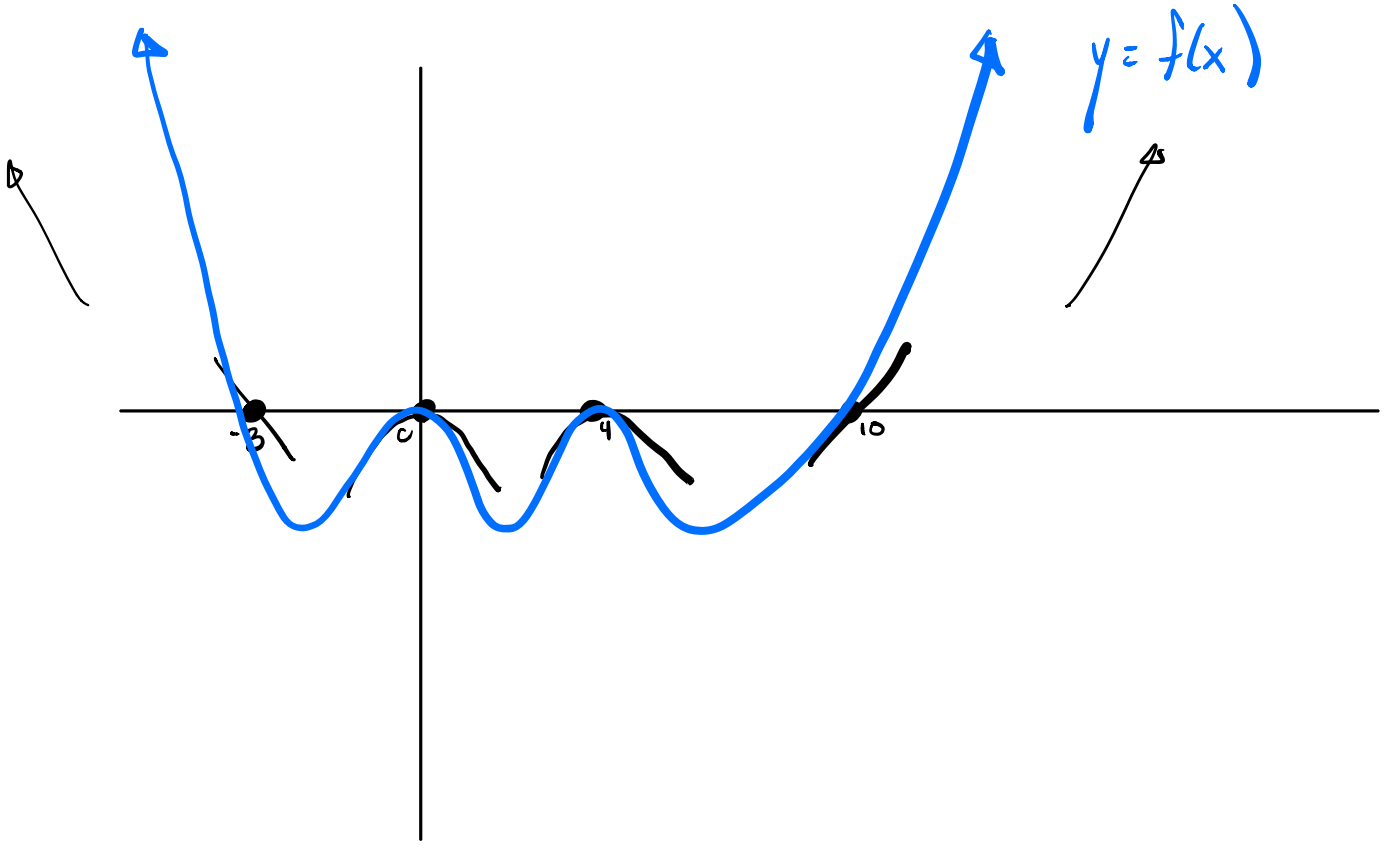
$x=0$ $x=-3$ $x=4$ $x=10$

GRAPH CROSSES X-AXIS AT THE ZEROS OF ODD DEGREE FACTORS (FACTORS RAISED TO ODD EXPONENTS)

THE SIGN OF THOSE FACTORS CHANGE

GRAPH BOUNCES OF X-AXIS AT THE ZEROS OF EVEN DEGREE FACTORS (FACTORS RAISED TO ODD EXPONENTS)

THE SIGN OF THOSE FACTORS DO NOT CHANGE.



THM.

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0}$$

← DEGREE n POLYNOMIAL WITH LEAD COEFFICIENT a_n

← DEGREE m POLYNOMIAL WITH LEAD COEFFICIENT b_m

$$= \begin{cases} \pm\infty & \text{IF } n > m \quad \left(+\infty \text{ IF } \frac{a_n}{b_m} > 0,\ -\infty \text{ IF } \frac{a_n}{b_m} < 0\right) \\ \frac{a_n}{b_m} & \text{IF } n = m \\ 0 & \text{IF } n < m \end{cases}$$

IDEA: WHEN $|x|$ IS LARGE, $x^n \gg x^{n-k}$ (MUCH BIGGER), $k = 1, 2, 3 \dots$

AND SO

$$\frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0} \approx \frac{a_n x^n}{b_m x^m}.$$

THUS

$$\lim_{x\to\infty} \frac{a_n x^n + a_{n-1}x^{n-1} + \dots + a_0}{b_m x^m + b_{m-1}x^{m-1} + \dots + b_0} = \frac{a_n}{b_m} \lim_{x\to\infty} x^{n-m}.$$

ex.

$$\lim_{x\to\infty} \frac{x^2 + 7x - 1}{-3x^3 + 2x - 3}$$

← DEG 2, LEAD COEFF 1

← DEG 3, LEAD COEFF -3

$$\approx \lim_{x\to\infty} \frac{x^2}{-3x^3} = -\frac{1}{3} \lim_{x\to\infty} \frac{1}{x} = -\frac{1}{3}\cdot 0 = 0$$

$$\lim_{x\to\infty} \frac{x^2+7x-1}{-3x^3+2x-3} = \lim_{x\to\infty} \frac{x^3\left(\frac{1}{x}+\frac{7}{x^2}-\frac{1}{x^3}\right)}{x^3\left(-3+\frac{2}{x^2}-\frac{3}{x^3}\right)}$$

$$= \frac{0}{-3} = 0.$$

ex. SKETCH $y = \dfrac{x^2(x-4)(x-6)^3}{(x+2)^3(x-6)^4}$

x-INT: $x=0$, $x=4$, $x=6$

VERTICAL ASYMPTOTES: ZEROS OF DENOM (THAT ARE NOT ALSO ZEROS OF NUMERATOR)

$x=-2$, $x=6$

$\lim_{x\to -2^-} y = \pm\infty$ $\lim_{x\to -2^+} y = \pm\infty$?

$\lim_{x\to 6^-} y = \pm\infty$ $\lim_{x\to 6^+} y = \pm\infty$.

LIMITS AT INFINITY: $\lim_{x\to\pm\infty} \dfrac{x^2(x-4)(x-6)^3}{(x+2)^3(x-6)^4} \approx \lim_{x\to\pm\infty} \dfrac{x^2(x)(x^3)}{x^3\,x^4}$

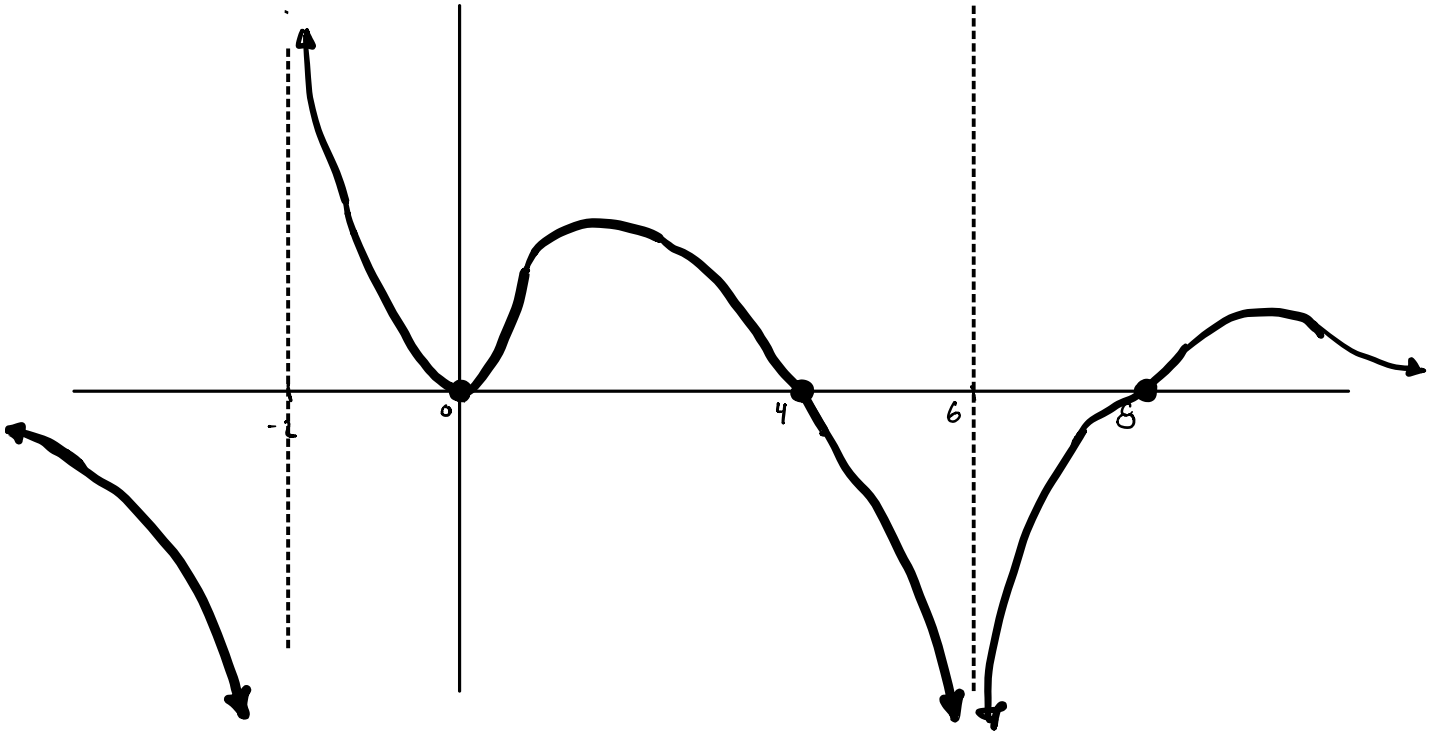
$\approx \lim_{x\to\pm\infty} \dfrac{x^6}{x^7} = \lim_{x\to\pm\infty} \dfrac{1}{x} = 0$

HORIZONTAL ASYMPTOTE: $y=0$

Note: $\lim_{x \to \infty} y \approx \lim_{x \to \infty} \frac{1}{x}$

As $x \to \infty$, $y \to 0^+$

$\lim_{x \to -\infty} y \approx \lim_{x \to -\infty} \frac{1}{x}$

As $x \to -\infty$, $y \to 0^-$

$$y = \frac{x^2 (x-4)(x-6)^3}{(x+2)^3 (x-6)^4}$$