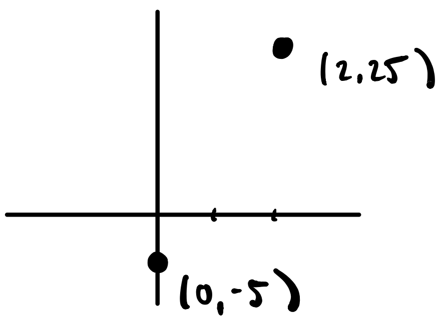
WRITTEN HW #1

Q6. $x^5 + 3x = x^3 + 5$. SHOW THAT THERE IS A SOL'N.

$$f(x) = x^5 - x^3 + 3x - 5 = 0$$

f IS A POLYNOMIAL $\Rightarrow$ CONTINUOUS ON DOMAIN $= (-\infty, \infty)$.



$f(0) = -5$

$f(2) = 2^5 - 2^3 + 3(2) - 5$
$= 32 - 8 + 6 - 5$
$= 25$

SINCE f IS CONTINUOUS ON $[0,2]$ AND $f(0) < 0$ & $f(2) > 0$, THE IVT IMPLIES THERE EXISTS A NUMBER c, $0 < c < 2$, SUCH THAT $f(c) = 0$, i.e. $c^5 + 3c = c^3 + 5$.

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§2.6 IMPLICIT DIFFERENTIATION

APPLICATION OF THE CHAIN RULE.

WARMUP:

ex. SUPPOSE f IS DIFFERENTIABLE. (f' EXISTS)

LET $y = f(x)$. LET $w = x^2 f(x) = x^2 y$

THEN $\frac{dw}{dx} = w' = \frac{d}{dx}[x^2] f(x) + x^2 \frac{d}{dx}[f(x)]$ PRODUCT RULE

$= 2x f(x) + x^2 f'(x)$

$= 2xy + x^2 y'$

ex. SUPPOSE f IS DIFFERENTIABLE. (f' EXISTS)

LET $y = f(x)$. LET $W = x^2 f(x)^3 = x^2 y^3$

$$\frac{dW}{dx} = W' = \frac{d}{dx}[x^2]\, f(x)^3 + x^2 \frac{d}{dx}\left[f(x)^3 \right]$$ PRODUCT RULE

$$= 2x f(x)^3 + x^2 \cdot 3f(x)^2 f'(x)$$ CHAIN RULE

$$= 2xy^3 + 3x^2y^2y'$$

IMPLICIT DIFFERENTIATION

CONSIDER THE EQUATION $x^2 + y^2 = 25$.

IT DESCRIBES A CURVE.

FIND EQ OF TANGENT LINE

AT THE POINT $(3,4)$.

IMPLICIT EQ.
NOT SOLVED FOR EITHER VARIABLE.
(y = ALL x's ; x = ALL y's
EXPLICIT)

NOTE: THE IMPLICIT EQ DEFINES y AS ONE OR MORE FUNCTIONS OF x.

$$y = \begin{cases} \sqrt{25 - x^2} & \text{IF } y \geq 0 \\ -\sqrt{25 - x^2} & \text{IF } y < 0 \end{cases}$$

EXPLICIT EQ'S

$$\frac{dy}{dx} = \begin{cases} \frac{1}{2}(25 - x^2)^{-1/2}(-2x) & \text{IF } y \geq 0 \\ -\frac{1}{2}(25 - x^2)^{-1/2}(-2x) & \text{IF } y < 0 \end{cases}.$$

Let's solve using Implicit Differentiation

(1) Assume that the implicit eq defines y as one or more differentiable functions of x, e.g. $y = f(x)$.

e.g. $x^2 + y^2 = 25 \longrightarrow x^2 + f(x)^2 = 25$

(2) Differentiate both sides of implicit eq:

$$\frac{d}{dx}\left[x^2 + y^2\right] = \frac{d}{dx}\left[25\right]$$

$$2x + 2y\,y' = 0$$

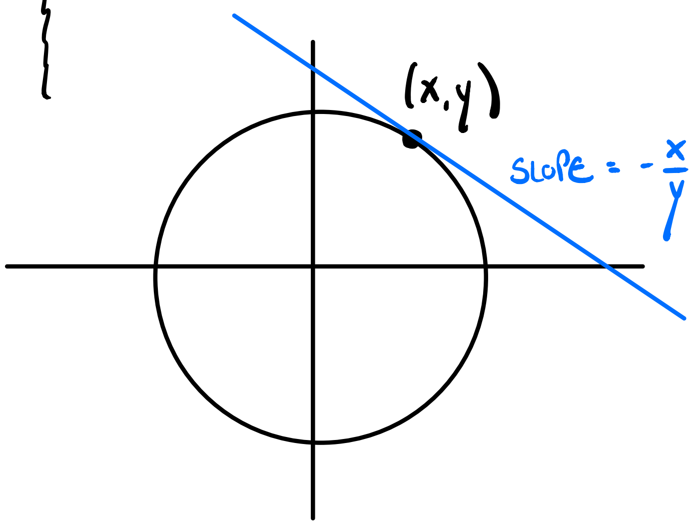
$$\frac{d}{dx}\left[x^2 + f(x)^2\right] = \frac{d}{dx}\left[25\right]$$

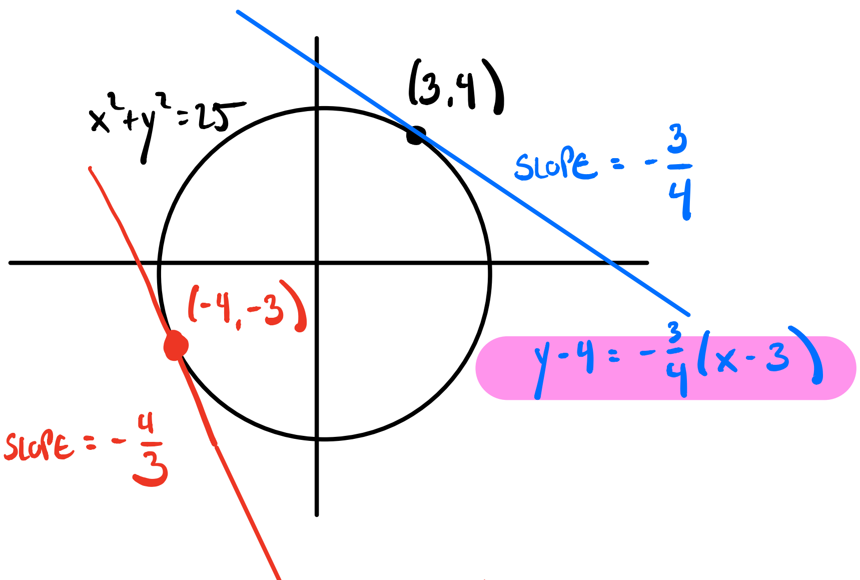
$$2x + 2f(x)f'(x) = 0$$

(3) Solve for y' $\left(\frac{dy}{dx}\right)$

$$2yy' = -2x$$

$$y' = -\frac{x}{y}$$





$$y + 3 = -\frac{4}{3}(x+4)$$

$$y - b = m(x - a)$$

Line through (a,b) with slope m

5–20 Find dy/dx by implicit differentiation.

5. $x^2 - 4xy + y^2 = 4$

6. $2x^2 + xy - y^2 = 2$

7. $x^4 + x^2y^2 + y^3 = 5$

8. $x^3 - xy^2 + y^3 = 1$

9. $\dfrac{x^2}{x + y} = y^2 + 1$

10. $y^5 + x^2y^3 = 1 + x^4y$

11. $y \cos x = x^2 + y^2$

12. $\cos(xy) = 1 + \sin y$

(7)

$$\frac{d}{dx}\left[x^4 + x^2y^2 + y^3\right] = \frac{d}{dx}\left[5\right]$$

$$4x^3 + 2xy^2 + \underbrace{x^2 \cdot 2y\frac{dy}{dx}}_{\text{CHAIN RULE.}} + \underbrace{3y^2\frac{dy}{dx}}_{\text{CHAIN RULE.}} = 0$$

WE ASSUME y IS A FUNCTION(S) OF x!

$$2x^2y\frac{dy}{dx} + 3y^2\frac{dy}{dx} = -4x^3 - 2xy^2$$

$$\frac{dy}{dx}\left(2x^2y + 3y^2\right) = -4x^3 - 2xy^2$$

$$\frac{dy}{dx} = \frac{-4x^3 - 2xy^2}{2x^2y + 3y^2}$$

IF WE PLUG IN A POINT (x, y) ON THE CURVE, THIS EXPRESSION GIVES THE SLOPE OF THE TANGENT LINE TO THE CURVE AT THAT POINT.

25–32 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

25. $y \sin 2x = x \cos 2y$, $(\pi/2, \pi/4)$

$$\frac{d}{dx}\left[y \sin(2x)\right] = \frac{d}{dx}\left[x \cos(2y)\right]$$

$$\frac{dy}{dx}\sin(2x) + y\cos(2x)\cdot 2 = 1\cos(2y) + x\left(-\sin(2y)\right)2\frac{dy}{dx}$$

$$\frac{dy}{dx}\sin(2x) + x\left(\sin(2y)\right)2\frac{dy}{dx} = 1\cos(2y) - y\cos(2x)\cdot 2$$

$$\frac{dy}{dx}\left[\sin(2x) + 2x\sin(2y)\right] = 1\cos(2y) - y\cos(2x)\cdot 2$$

$$\frac{dy}{dx} = \frac{1\cos(2y) - y\cos(2x)\cdot 2}{\sin(2x) + 2x\sin(2y)}$$

AT THE POINT $\left(\frac{\pi}{2}, \frac{\pi}{4}\right)$, SLOPE OF THE TANGENT LINE TO THE CURVE IS

$$\left.\frac{dy}{dx}\right|_{\left(\frac{\pi}{2}, \frac{\pi}{4}\right)} = \frac{1\cos\left(\frac{\pi}{2}\right) - \frac{\pi}{4}\cos(\pi)\cdot 2}{\sin(\pi) + 2\frac{\pi}{2}\sin\left(\frac{\pi}{2}\right)}$$

$$= \frac{0 + \frac{\pi}{2}}{0 + \pi} = \frac{\pi\left(\frac{1}{2}\right)}{\pi} = \frac{1}{2}$$

$$y - \frac{\pi}{4} = m\left(x - \frac{\pi}{2}\right) \qquad \boxed{y - \frac{\pi}{4} = \frac{1}{2}\left(x - \frac{\pi}{2}\right)}$$

34. (a) The curve with equation $y^2 = x^3 + 3x^2$ is called the **Tschirnhausen cubic**. Find an equation of the tangent line to this curve at the point $(1, -2)$.

(b) At what points does this curve have horizontal tangents?

(c) Illustrate parts (a) and (b) by graphing the curve and the tangent lines on a common screen.

$$\frac{d}{dx}\left[y^2\right] = \frac{d}{dx}\left[x^3 + 3x^2\right]$$

$$2y\frac{dy}{dx} = 3x^2 + 6x = 3x(x+2)$$

$$\underbrace{\frac{dy}{dx} = \frac{3x(x+2)}{2y} = 0}_{\text{HORIZONTAL TANGENT} \Rightarrow \frac{dy}{dx} = 0}$$

$x = 0$

$\Rightarrow y^2 = 0$

$\Rightarrow y = 0$

$\Rightarrow \frac{dy}{dx}$ UNDEFINED.

$x = -2$

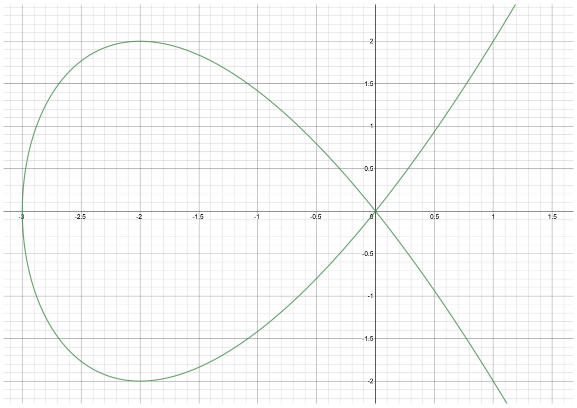
$\Rightarrow y^2 = (-2)^3 + 3(-2)^2$

$= -8 + 12$

$y^2 = 4$

$y = \pm 2$

$(-2, 2)$, $(-2, -2)$



-3
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
1.5
1
0.5
-0.5
-1
-1.5
-2