Math 1206 - Midterm Exam Review - 10/9/20

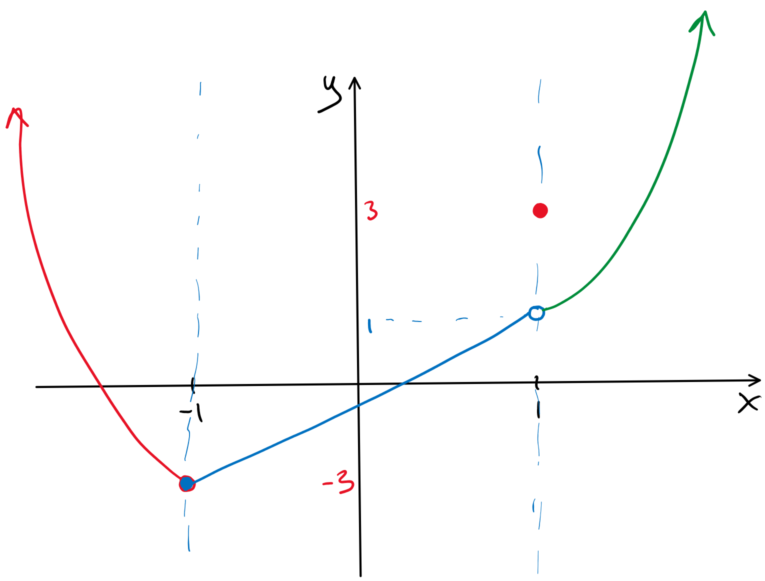
1. CONSIDER THE FUNCTION

$$f(x) = \begin{cases} x^2 & \text{IF } 1 < x \\ 3 & \text{IF } x = 1 \\ 2x - 1 & \text{IF } -1 \leq x < 1 \\ (x+2)^2 - 4 & \text{IF } x < -1 \end{cases}$$

plug in -1

$(-1+2)^2 - 4 = -3$

(a.) SKETCH THE GRAPH $y = f(x)$.



(b) FIND $\lim_{x \to 1} f(x)$ OR SHOW IT D.N.E.

$\lim_{x \to 1} f(x) = 1$ (from the graph).

(c) IS f CONTINUOUS AT $x = 1$?

f is continuous at $\lim_{x \to 1} f(x) = f(1)$

$1 \neq 3$

$\Rightarrow \boxed{f \text{ is } \underline{\text{not}} \text{ continuous at } x = 1}$

(d) SHOW THAT f IS CONTINUOUS AT $x=-1$
(DEF. OF CONTINUOUS)

We need to show $\lim_{x\to -1} f(x) = f(-1)$

① The limit must exist at $x=a$
(both one sided limits must exist and be equal)

② The function must be defined at $x=a$

③ The values in ① and ② must be equal.

① $\lim_{x\to -1} f(x)$

Need to check the one sided limits since defn of f changes.

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-} \left((x+2)^2 - 4\right)$$
$$= (-1+2)^2 - 4$$
$$= 1 - 4$$
$$= -3$$

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} (2x-1)$$
$$= 2(-1) - 1$$
$$= -3$$

$$\Rightarrow \lim_{x\to -1} f(x) = -3$$

② $f(-1)$

$$f(-1) = 2(-1) - 1$$
$$= -3$$

③ Since parts ① and ② are the same,

$$\boxed{f(x) \text{ is continuous at } x=-1}$$

(e) USE DEFINITION OF DERIVATIVE AS A LIMIT TO SHOW THAT f IS DIFFERENTIABLE AT $x=-1$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Show that f is differentiable at $x = -1$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: we need one sided limits here, since f changes on either side of $x = -1$

On the left

$$\lim_{h \to 0^-} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0^-} \frac{[((-1+h)+2)^2 - 4] - (-3)}{h}$$

$$= \lim_{h \to 0^-} \frac{(1+h)^2 - 4 + 3}{h}$$

$$= \lim_{h \to 0^-} \frac{\cancel{1} + 2h + h^2 - \cancel{4} + \cancel{3}}{h}$$

$$= \lim_{h \to 0^-} \frac{\cancel{h}(2+h)}{\cancel{h}}$$

$$= 2$$

On the right

$$\lim_{h \to 0^+} \frac{f(-1+h) - f(-1)}{h}$$

$$= \lim_{h \to 0^+} \frac{[2(-1+h) - 1] - (-3)}{h}$$

$$= \lim_{h \to 0^+} \frac{-\cancel{2} + 2h - \cancel{1} + \cancel{3}}{h}$$

$$= \lim_{h \to 0^+} \frac{2\cancel{h}}{\cancel{h}}$$

$$= 2$$

$$\Rightarrow \lim_{h \to 0} \frac{f(-1+h) - f(-1)}{h} = 2 = f'(-1)$$

So, $\boxed{f' \text{ exists at } x = -1}$

(Alternatively, we can use "x+h" and "x" instead of "-1+h" and "-1" then plug in $x = -1$).

So, for e.g. $\lim_{h \to 0^-} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0^-} \frac{[(x+h+2)^2 - 4] - [(x+2)^2 - 4]}{h}$$

$(x+h+2)(x+h+2)$

$$= \lim_{h \to 0^-} \frac{\cancel{x^2} + \cancel{xh} + \cancel{2x} + \cancel{xh} + \cancel{h^2} + \cancel{2h} + \cancel{2x} + \cancel{2h} + \cancel{4} - \cancel{4} - (\cancel{x^2} + \cancel{4x} + \cancel{4} - \cancel{4})}{h}$$

$$= \lim_{h \to 0^-} \frac{2xh + h^2 + 4h}{h}$$

$$\lim_{h\to 0^-} \frac{\qquad}{h}$$

$$= \lim_{h\to 0^-} \frac{\cancel{h}(2x+h+4)}{\cancel{h}}$$

$$= 2x+4$$

$$\Rightarrow f'(x)^- = (2x+4)$$

$$\Rightarrow f'(-1)^- = \boxed{2}$$

2. Use Squeeze Thm to Calculate

$$\lim_{x\to 0} \boxed{1 + x^2 \sin\left(\frac{2\pi}{\sqrt{|x|}}\right)}$$

If $x \neq 0$: $\boxed{-1 \leq \sin\left(\frac{2\pi}{\sqrt{|x|}}\right) \leq 1}$ → Start with the part that has bounds!

$$\Rightarrow -x^2 \leq x^2 \sin\left(\frac{2\pi}{\sqrt{|x|}}\right) \leq x^2$$

$$\Rightarrow 1 - x^2 \leq \boxed{1 + x^2 \sin\left(\frac{2\pi}{\sqrt{|x|}}\right)} \leq 1 + x^2$$

→ We built up to the function we care about!

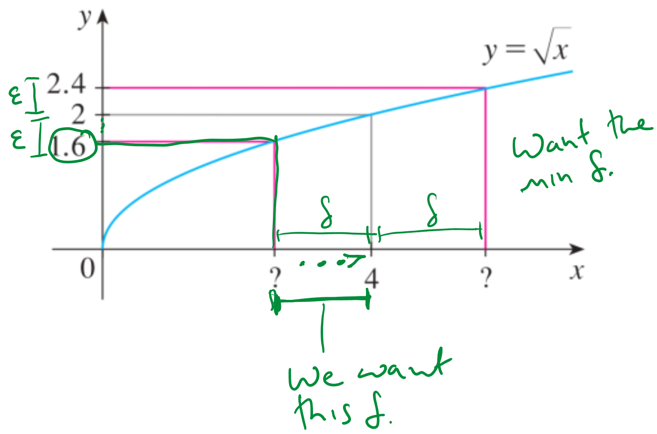
$$\Rightarrow \lim_{x\to 0}(1-x^2) \leq \lim_{x\to 0}\left(1 + x^2 \sin\left(\frac{2\pi}{\sqrt{|x|}}\right)\right) \leq \lim_{x\to 0}(1+x^2)$$

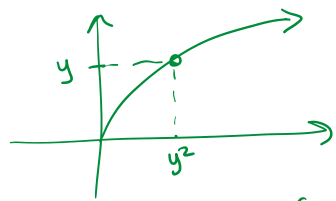
$$\Rightarrow 1 \leq \lim_{x\to 0}\left(1 + x^2 \sin\left(\frac{2\pi}{\sqrt{|x|}}\right)\right) \leq 1$$

$$\Rightarrow \boxed{\lim_{x\to 0}\left(1 + x^2 \sin\left(\frac{2\pi}{\sqrt{|x|}}\right)\right) = 1 \quad \text{by Sq. Thm!}}$$

3. Use the given graph of $f(x) = \sqrt{x}$ to find a number δ such that

$$\text{if} \quad |x - 4| < \delta \quad \text{then} \quad |\sqrt{x} - 2| < 0.4$$





The ? here $= (1.6)^2$

So $\boxed{\delta = 4 - (1.6)^2}$ or anything smaller.

4. (a.) STATE THE INTERMEDIATE VALUE THEOREM.

Memorize!!!!

10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

(b.) SHOW THAT THE EQUATION $\dfrac{1}{3 - \sqrt{x}} = \cos(\pi x)$ HAS AT LEAST ONE SOLUTION.

To say $\dfrac{1}{3-\sqrt{x}} = \cos \pi x$ is the same as

saying: $\cos \pi x - \dfrac{1}{3-\sqrt{x}} = 0$

Set $f(x) = \cos \pi x - \dfrac{1}{3-\sqrt{x}}$

$f(x) = 0$ at some pt.

Set $f(x) = \cos \pi x - \frac{1}{3-\sqrt{x}}$

We want to show, $f(x) = 0$ at some pt.

① f is <u>continuous</u> (on $[0,9)\cup(9,\infty)$).

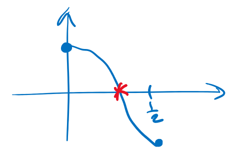
In particular, f is continuous on $[0, \frac{1}{2}]$.

$$f(0) = \cos 0 - \frac{1}{3-0} = 1 - \frac{1}{3} = \frac{2}{3} > 0.$$

$$f(\tfrac{1}{2}) = \cos \frac{\pi}{2} - \frac{1}{3-\sqrt{1/2}} = 0 - \frac{1}{3-\sqrt{1/2}} < 0$$

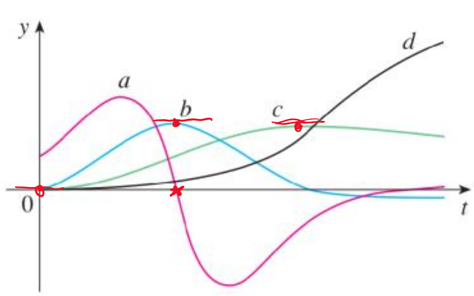
$f(1) = \cos \pi - \frac{1}{3-\sqrt{1}} = -1 - \frac{1}{2} = -\frac{3}{2} < 0$ also works!

So, by the IVT, there is some $c \in (0, 1/2)$ such that $f(c) = 0$.



Setting $x = c$ is our soln.

5. The figure shows the graphs of four functions. One is the position function of a car, one is the velocity of the car, one is its acceleration, and one is its jerk. Identify each curve, and explain your choices.



Note: $a = b'$

$b = c'$

$c = d'$

... position

So we have:

$$\boxed{\begin{aligned} d &= \text{function} = \text{position} \\ c &= d' = \text{velocity} \\ b &= c' = d'' = \text{acceleration} \\ a &= b' = d''' = \text{jerk} \end{aligned}}$$

6. CALCULATE THE LIMIT OR SHOW THAT IT DOES NOT EXIST.

$$\lim_{x \to 0} \sqrt[3]{\frac{\sin(3x)}{\sin(5x)}}$$

Special trig limits

1. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 = \lim_{\theta \to 0} \frac{\theta}{\sin\theta}$

2. $\lim_{\theta \to 0} \frac{\tan\theta}{\theta} = \lim_{\theta \to 0} \frac{\theta}{\tan\theta} = 1$

3. $\lim_{\theta \to 0} \frac{1-\cos\theta}{\theta} = \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

Also recall: "limits pass through continuous functions"

i.e. $\lim_{x \to a} f(x) = f\left(\lim_{x \to a} x\right) = f(a)$

if f is continuous!

$$\lim_{x\to 0} \sqrt[3]{\frac{\sin(3x)}{\sin(5x)}} = \sqrt[3]{\lim_{x\to 0} \frac{\sin(3x)}{\sin(5x)}}$$

$$= \sqrt[3]{\lim_{x\to 0} \frac{3x}{3x}\frac{\sin(3x)}{1} \cdot \frac{1}{\sin(5x)} \cdot \frac{5x}{5x}}$$

$$= \sqrt[3]{\lim_{x\to 0}\left(3\cancel{x} \cdot \frac{\sin(3x)}{3x} + \frac{1}{5\cancel{x}} \cdot \frac{5x}{\sin(5x)}\right)}$$

(since limits distribute across sums and products when finite)

$$= \sqrt[3]{\lim_{x\to 0}\left(\frac{3}{5} \cdot \underbrace{\frac{\sin 3x}{3x}}_{1} \cdot \underbrace{\frac{5x}{\sin 5x}}_{1}\right)}$$

$$= \boxed{\sqrt[3]{\frac{3}{5}}}$$

7. FIND AN EQUATION FOR THE TANGENT LINE TO THE CURVE

$$y = \frac{x^2 \cos x}{\sqrt{4 + \sin x}}$$

AT THE POINT $\left(\pi, \frac{-\pi^2}{2}\right)$.

Recall: $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$

$$\Rightarrow y' = \frac{(4+\sin x)^{1/2}\left[2x\cos x - x^2\sin x\right] - x^2\cos x \cdot \frac{1}{2}(4+\sin x)^{-1/2}(\cos x)}{4+\sin x}$$

$$= \frac{x\cdot(4+\sin x)^{-1/2}\left[(4+\sin x)(2\cos x - x\sin x) - \frac{1}{2}x\cos^2 x\right]}{4+\sin x}$$

$$= \frac{x\left((4+\sin x)(2\cos x - x\sin x) - \frac{1}{2}x\cos^2 x\right)}{(4+\sin x)^{3/2}}$$

$$m = y'(\pi) = \frac{\pi\left[(4+\cancelto{0}{\sin\pi})(2\cancelto{-1}{\cos\pi} - \pi\cancelto{0}{\sin\pi}) - \frac{1}{2}\pi\cancelto{1}{(\cos\pi)^2}\right]}{(4+\cancelto{0}{\sin\pi})^{3/2}}$$

$$= \frac{\pi\left(4(-2) - \frac{\pi}{2}\right)}{4^{3/2}}$$

$$= \frac{-8\pi - \frac{\pi^2}{2}}{8} \cdot \frac{2}{2}$$

$$m = \frac{-16\pi - \pi^2}{16}$$

point $= (x_1, y_1) = \left(\pi, -\frac{\pi^2}{2}\right)$

$\Rightarrow$ Tangent line: $y - y_1 = m(x - x_1)$

$$\Rightarrow \boxed{y + \frac{\pi^2}{2} = \left(\frac{-16\pi - \pi^2}{16}\right)(x - \pi)}$$

8. FIND $\frac{dy}{dx}$ USING IMPLICIT DIFFERENTIATION.

$$\sin(3x - 8y) = x^3 y^4$$

Idea: Specifically as it regards to $f(x,y) = c$ and you want to find $\frac{dy}{dx}$.

1. Go through, differentiate as usual, obeying all rules
2. BUT whenever you differentiate a y-term, multiply by y'. (Chain rule).

i.e. Multiply by a prime if you're not differentiating the independent variable.

$$\sin(3x-8y) = \underbrace{x^3 y^4}_{\text{product rule!}}$$

$$\Rightarrow \cos(3x-8y)(3-8y') = 3x^2 \cdot y^4 + x^3 \cdot 4y^3 \cdot y'$$

Now we need to solve for the y':

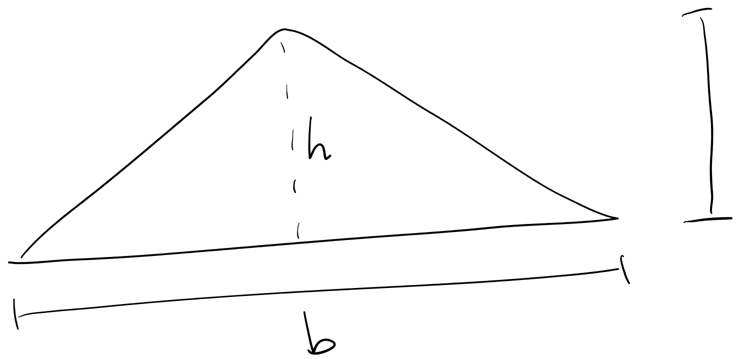
$$\Rightarrow 3\cos(3x-8y) - 8y'\cos(3x-8y) = 3x^2y^4 + 4x^3y^3y'$$

$$\Rightarrow \underbrace{3\cos(3x-8y) - 3x^2y^4}_{\text{all the stuff w/ no } y'} = \underbrace{4x^3y^3y' + 8\cos(3x-8y)y'}_{\text{all the stuff with } y'}$$

$$\Rightarrow 3\cos(3x-8y) - 3x^2y^4 = (4x^3y^3 + 8\cos(3x-8y))y'$$

$$\Rightarrow \boxed{\frac{3\cos(3x-8y) - 3x^2y^4}{4x^3y^3 + 8\cos(3x-8y)} = y'}$$

9. The altitude of a triangle is increasing at a rate of 1 cm/min while the area of the triangle is increasing at a rate of 2 cm^2/min. At what rate is the base of the triangle changing when the altitude is 10 cm and the area is 100 cm^2?



Let A = area

Given: $\frac{dh}{dt} = +1$ (increasing)

$\frac{dA}{dt} = 2$

Want: $\frac{db}{dt}$ when $h = 10$ and $A = 100$

Equation: $A = \underline{\frac{1}{2} b} \, \underline{h}$

(equation that relates all variables)

Differentiate w.r.t. t: $\frac{dA}{dt} = \frac{1}{2} \cdot \frac{db}{dt} \cdot h + \frac{1}{2} b \cdot \frac{dh}{dt}$

Plug in knowns solve for unknown: $2 = \frac{1}{2}\left(\frac{db}{dt}\right)(10) + \frac{1}{2}(b)(1)$

If there are unknowns,

solve fo

If there are more unknowns, go back to equation.

$A = \frac{1}{2} b h$

$\Rightarrow$ when $A = 100$, $h = 10$

$\Rightarrow 100 = \frac{1}{2} b (10)$

$\Rightarrow b = 20$

$$\Rightarrow 2 = \frac{1}{2}\left(\frac{db}{dt}\right)(10) + \frac{1}{2}(20)(1)$$

$$\Rightarrow 2 = 5\frac{db}{dt} + 10$$

$$\Rightarrow \boxed{-\frac{8}{5} = \frac{db}{dt}}$$

10. FIND THE LINEAR APPROXIMATION TO

$f(x) = \sqrt[3]{x}$ AT $x = 64 \rightarrow a$,

AND USE THIS TO APPROXIMATE $\sqrt[3]{63} \rightarrow x$.

The linearization (or linear approx.) of $f(x)$ at $x = a$ is:

$$L(x) = f(a) + f'(a)(x-a)$$

$$L(x) = f(a) + f'(a)(x-a)$$

"x" as the ugly value, "a" is the nice value.

$$f(x) = \sqrt[3]{x}$$

$$\Rightarrow f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x}^{\,2}}$$

$$\Rightarrow f(a) = f(64) = \sqrt[3]{64} = 4$$

$$\Rightarrow f'(a) = f'(64) = \frac{1}{3\sqrt[3]{64}^{\,2}} = \frac{1}{48}$$

$\Rightarrow$ linear approx of f at $x=64$ is:

$$L(x) = 4 + \frac{1}{48}(x-64)$$

$$\Rightarrow \sqrt[3]{63} \approx L(63)$$

$$= 4 + \frac{1}{48}(63-64)$$

$$= \boxed{4 - \frac{1}{48}} \approx 3.979167$$

tual $\approx 3.97905720\ldots$

(For curiosity: actual $\approx 3.97905720\ldots$)