§ 3.3 How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

PROOF: TAKE $a, b \in I$ WITH $a < b$.

MVT. $\Rightarrow$ $f(b) - f(a) = f'(c)\underbrace{(b-a)}_{\text{POSITIVE}}$ FOR SOME $c \in (a,b)$.

IF $f'(c) > 0$ THEN $f(b) - f(a) > 0$, i.e. $f(b) > f(a)$.

IF $f'(c) < 0$ THEN $f(b) - f(a) < 0$, i.e. $f(b) < f(a)$.

SINCE THIS HOLDS FOR ALL $a < b$ IN THE INTERVAL I,

f IS INCREASING / DECREASING ON I. $\square$

ex. FIND THE INTERVAL(S) ON WHICH

$$f(x) = x^4 - 4x^3 + 4x^2$$

IS INCREASING / DECREASING.

FIND THE INTERVAL(S) ON WHICH

$f'(x)$ IS POS / NEG

$f'(x) = 4x^3 - 12x^2 + 8x$ ← f' IS POLYNOMIAL $\Rightarrow$ CONTINUOUS ON $\mathbb{R}$

IF f' WERE TO GO FROM POS → NEG OR NEG → POS

IT WOULD HAVE TO PASS THROUGH 0.

INT. VALUE THM.

① FIND WHERE $f'(x) = 0$. THESE ZEROS BREAK UP $\mathbb{R}$ INTO INTERVALS ON WHICH THE SIGN OF f' DOES NOT CHANGE.

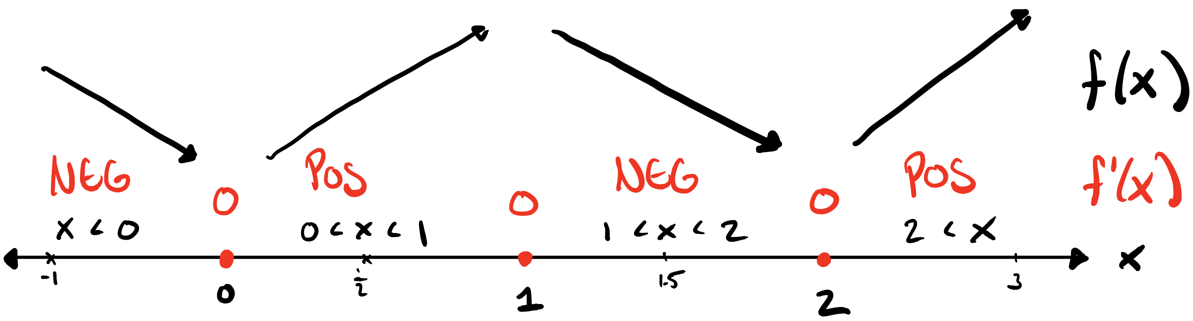
② DETERMINE THE SIGN OF f' ON EACH OF THESE INTERVALS.

$$f'(x) = 4x^3 - 12x^2 + 8x = 0$$

$$4x(x^2 - 3x + 2) = 0$$

$$4x(x-2)(x-1) = 0$$

$x = 0 \qquad x = 2 \qquad x = 1$



$4x$	NEG	POS	POS	POS
$(x-2)$	NEG	NEG	NEG	POS
$(x-1)$	NEG	NEG	POS	POS
$f'(x) = 4x(x-2)(x-1)$	NEG	POS	NEG	POS

$f'(x)$ IS POSISITIVE ON $(0,1) \cup (2,\infty)$

$f'(x)$ IS NEGATIVE ON $(-\infty,0) \cup (1,2)$

$\downarrow$

$f(x)$ IS INCREASING ON $(0,1) \cup (2,\infty)$

$f(x)$ IS DECREASING ON $(-\infty,0) \cup (1,2)$

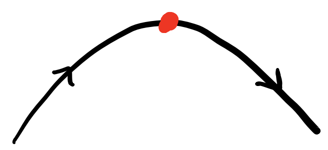
The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

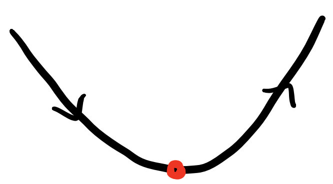
(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' is positive to the left and right of c, or negative to the left and right of c, then f has no local maximum or minimum at c.

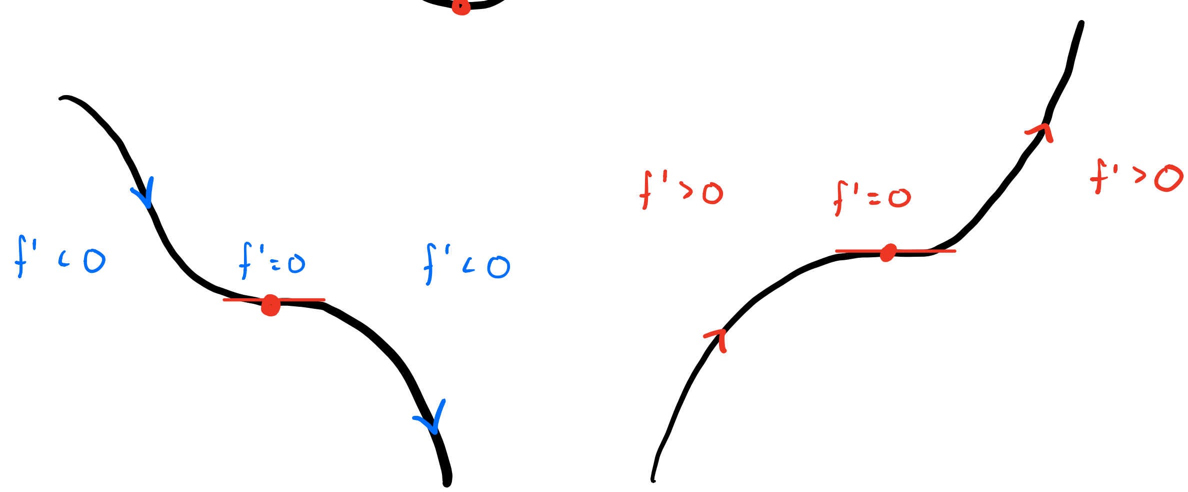
(a) f CHANGES FROM INCR. TO DECR $\Rightarrow$ LOCAL MAX



(a) f CHANGES FROM DECR. TO INCR $\Rightarrow$ LOCAL MIN



(c)



ex. FIND ALL LOCAL MAX/MIN'S OF

$$f(x) = x^{2/3}(6-x)^{1/3} \qquad \mathrm{Dom}(f) = \mathbb{R}$$

FERMAT'S THM: LOCAL MAX/MINS OCCUR ONLY AT CRITICAL PNTS.

1. FIND CRIT. PNTS $\left(x \in \mathrm{Dom}(f) \ \& \ f'(x) = \begin{cases} 0 \\ \text{UND} \end{cases} \right)$
2. USE 1ST DERIV. TEST TO CLASSIFY EACH AS EITHER LOCAL MAX, LOCAL MIN, OR NEITHER.

① $$f'(x) = \frac{2}{3}x^{-1/3}(6-x)^{1/3} + x^{2/3}\frac{1}{3}(6-x)^{-2/3}(-1) = 0$$

$$= \underbrace{\frac{1}{3}x^{-1/3}(6-x)^{-2/3}}_{\text{GCD}}\left[2(6-x) - x\right] = 0 \text{ OR UND.}$$

GCD = PRODUCT OF COMMON FACTORS RAISED TO LOWEST APPEARING EXPONENTS

$$\frac{12-3x}{3x^{1/3}(6-x)^{2/3}} = \frac{4-x}{x^{1/3}(6-x)^{2/3}} = 0$$

$$\frac{1}{3}\cdot\frac{1}{x^{1/3}}\cdot\frac{1}{(6-x)^{2/3}}\left[12-3x\right]$$

$f'(x) = 0$ WHEN $x = 4$

$f'(x)$ IS UNDEFINED WHEN $x = 6, 0$

	$x<0$	0	$0<x<4$	4	$4<x<6$	6	$x>6$
f	$\searrow$		$\nearrow$		$\searrow$		$\searrow$
f'	NEG	UND	POS	0	NEG	UND	NEG
$4-x$	POS		POS		NEG		NEG
$x^{1/3}$	NEG		POS		POS		POS
$(6-x)^{2/3}$	POS		POS		POS		POS

$$\left[(6-x)^{1/3}\right]^2 \geq 0$$

AT $x=0$, f' GOES FROM NEG $\to$ POS

f GOES FROM DECR $\to$ INCR

$\therefore$ $x=0$ IS LOCAL MIN

$f(0)=0$ = LOCAL MIN VALUE.

AT $x=4$, f' GOES FROM POS TO NEG

f GOES FROM INCR TO DECR

$\therefore$ $x=4$ IS A LOCAL MAX

$$f(4) = 4^{2/3}(6-4)^{1/3} = 4^{2/3}\,2^{1/3}$$
$$= (2^2)^{2/3}\,2^{1/3}$$
$$= 2^{4/3}\,2^{1/3} = 2^{5/3}$$

LOCAL MAX VALUE

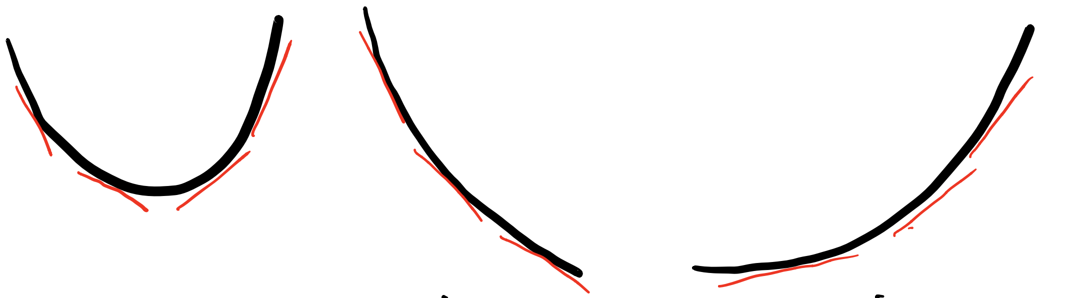
AT $x=6$, f' GOES FROM NEG TO NEG

f GOES FROM DECR TO DECR

$\therefore$ $x=6$ IS NEITHER LOCAL MAX OR LOCAL MIN.

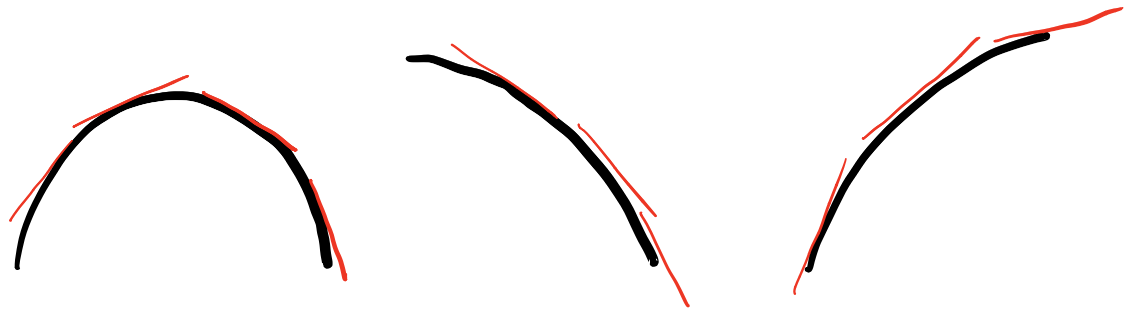
Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

CONCAVE UP



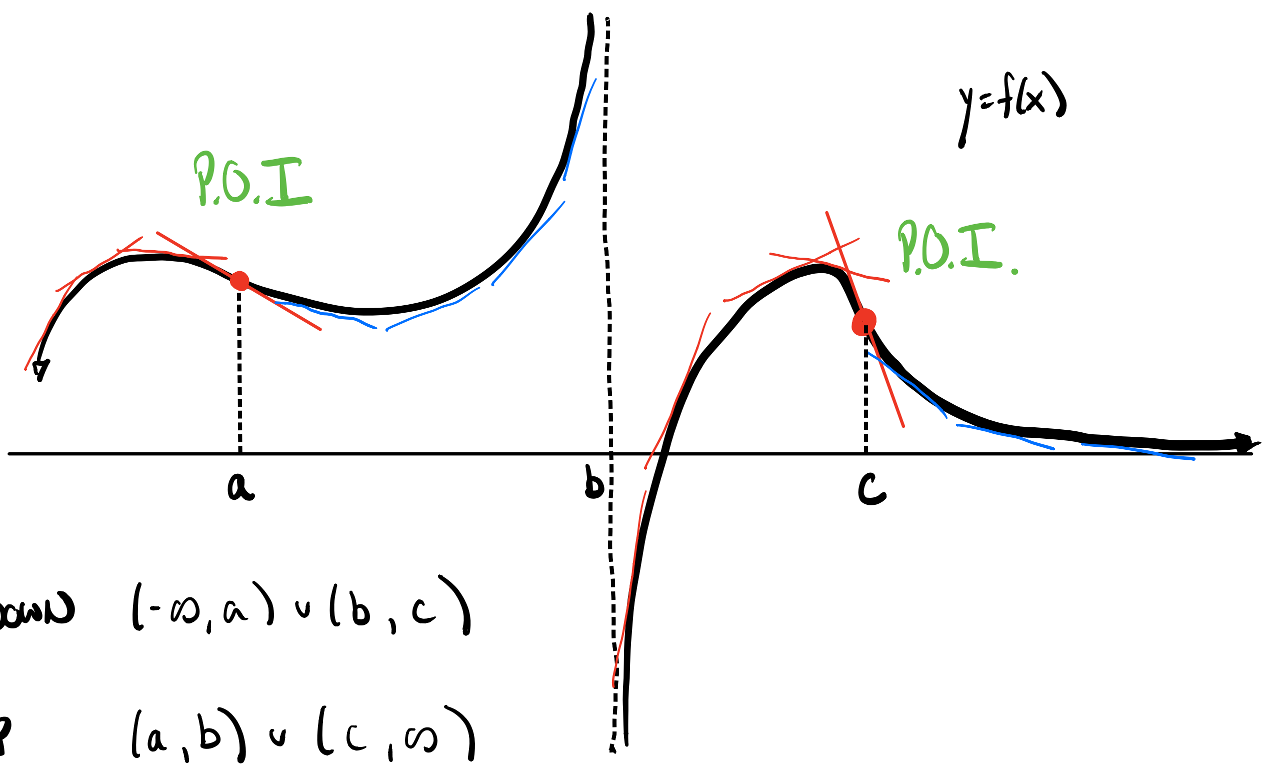
DECR. & CONC. UP.   INCR. & CONC. UP

CONCAVE DOWN



DECR. & CONC. DOWN   INCR. & CONC. DOWN

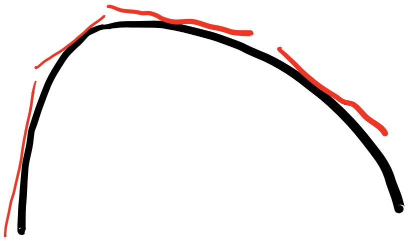
ex



CONCAVE DOWN $(-\infty, a) \cup (b, c)$

CONCAVE UP $(a, b) \cup (c, \infty)$

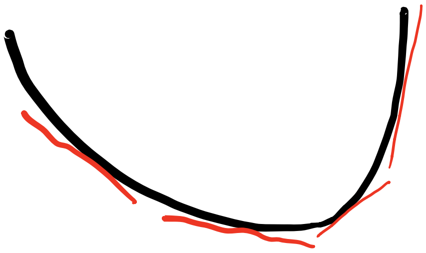
Definition A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.



CONCAVE DOWN: SLOPE IS DECREASING

f' IS DECREASING

f'' IS NEGATIVE



CONCAVE UP: SLOPE IS INCREASING

f' IS INCREASING

f'' IS POSITIVE.

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

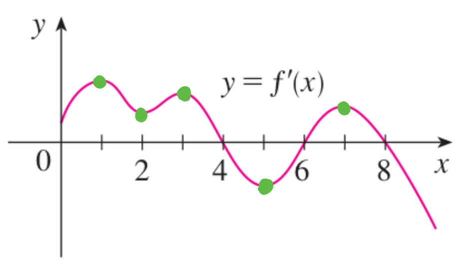
8. The graph of the first derivative f' of a function f is shown.

(a) On what intervals is f increasing? Explain.

(b) At what values of x does f have a local maximum or minimum? Explain.

(c) On what intervals is f concave upward or concave downward? Explain.

(d) What are the x-coordinates of the inflection points of f? Why?



(a) $f' > 0$:

$(0,4) \cup (6,8)$

(b) LOCAL MAX @ $x = 4, 8$

(f': POS → NEG)

LOCAL MIN @ $x = 6$

(f': NEG → POS)

(c) CONCAVE UP ($f'' > 0$, f' INCR) : $(0,1) \cup (2,3) \cup (5,7)$

CONCAVE DOWN ($f'' < 0$, f' DECR) : $(1,2) \cup (3,5) \cup (7,9)$

(d) P.O.I. WHEN f'' CHANGES FROM NEG → POS

OR POS → NEG

WHEN f CHANGES FROM DECR → INCR

OR INCR → DECR.

$x = 1, 2, 3, 5, 7.$