§ 3.7 Optimization Problems

Finding MAX/MIN

Steps In Solving Optimization Problems

1. **Understand the Problem** The first step is to read the problem carefully until it is clearly understood. Ask yourself: What is the unknown? What are the given quantities? What are the given conditions? Domain? Closed interval?
2. **Draw a Diagram** In most problems it is useful to draw a diagram and identify the given and required quantities on the diagram.
3. **Introduce Notation** Assign a symbol to the quantity that is to be maximized or minimized (let's call it Q for now). Also select symbols $(a, b, c, \ldots, x, y)$ for other unknown quantities and label the diagram with these symbols. It may help to use initials as suggestive symbols—for example, A for area, h for height, t for time.
4. Express Q in terms of some of the other symbols from Step 3.
5. If Q has been expressed as a function of more than one variable in Step 4, use the given information to find relationships (in the form of equations) among these variables. Then use these equations to eliminate all but one of the variables in the expression for Q. Thus Q will be expressed as a function of *one* variable x, say, $Q = f(x)$. Write the domain of this function in the given context.
6. Use the methods of Sections 3.1 and 3.3 to find the *absolute* maximum or minimum value of f. In particular, if the domain of f is a closed interval, then the Closed Interval Method in Section 3.1 can be used.

12. Consider the following problem: A box with an open top is to be constructed from a square piece of cardboard, 3 ft wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(a) Draw several diagrams to illustrate the situation, some short boxes with large bases and some tall boxes with small bases. Find the volumes of several such boxes. Does it appear that there is a maximum volume? If so, estimate it.

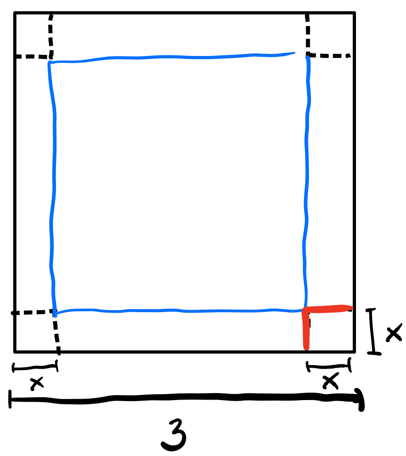
(b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with your symbols.

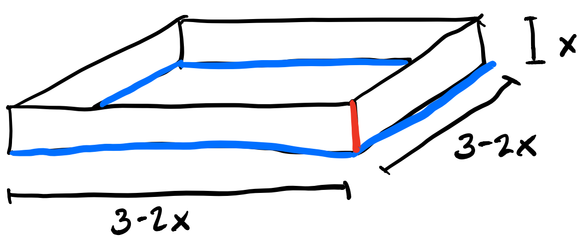
(c) Write an expression for the volume.

(d) Use the given information to write an equation that relates the variables.

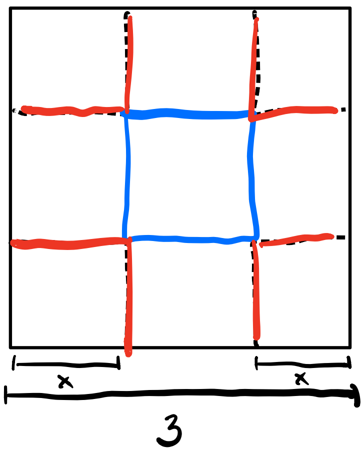
(e) Use part (d) to write the volume as a function of one variable.

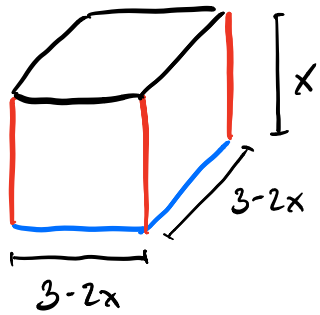
(f) Finish solving the problem and compare the answer with your estimate in part (a).

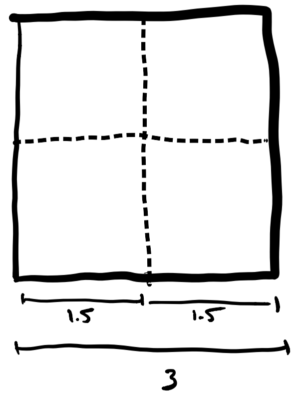




$$V = lwh = (3-2x)^2 x$$





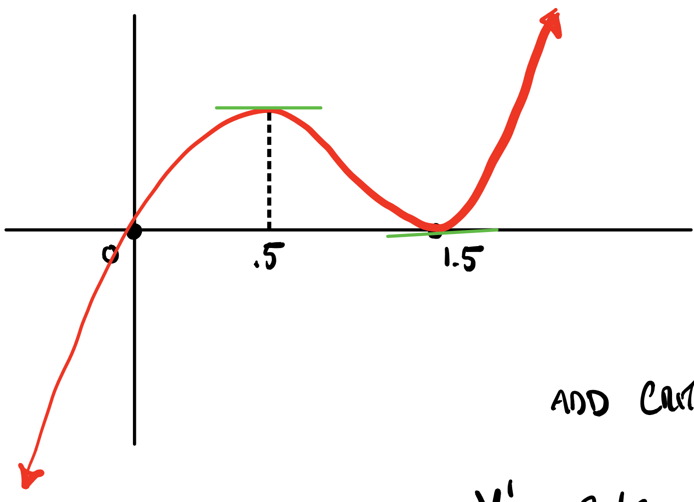


DOMAIN: $0 \le x \le 1.5$ — CLOSED, V CONTINUOUS

$\Rightarrow \exists$ ABS MAX / MIN

FIND ABS. MAX OF $V = (3-2x)^2 x$ ON $[0, 1.5]$

$V = 4x^3 + \dots$



	x	$V = (3-2x)^2 x$
ENDPOINTS	0	0
	1.5	0
	0.5	2

ABS MAX.

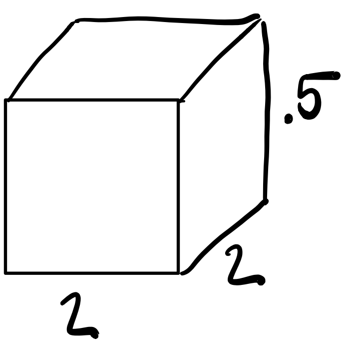
ADD CRITICAL POINTS:

$$V' = 2(3-2x)(-2)x + (3-2x)^2 = 0$$

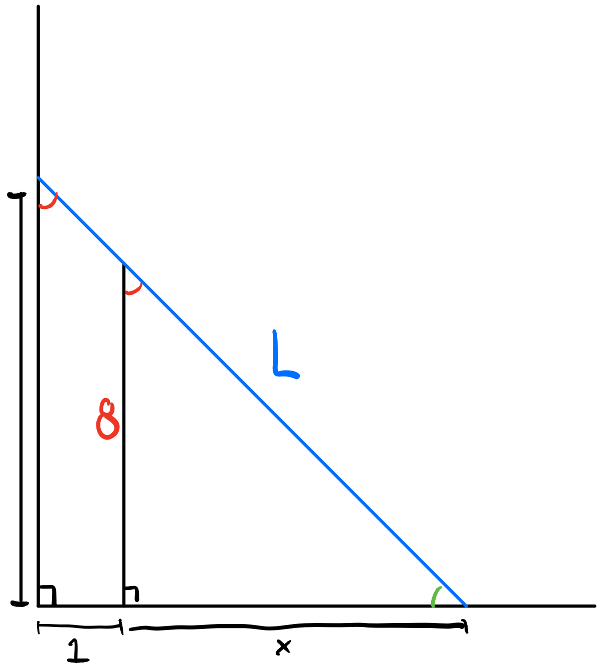
$$(3-2x)(-4x + (3-2x)) = 0$$

$$(3-2x)(-6x+3) = 0$$

$x = 1.5$ $x = \frac{1}{2}$



ex. A WALL 8 ft HIGH IS 1 ft FROM A HOUSE. FIND THE LENGTH L OF THE SHORTEST LADDER OVER THE WALL TO THE HOUSE.



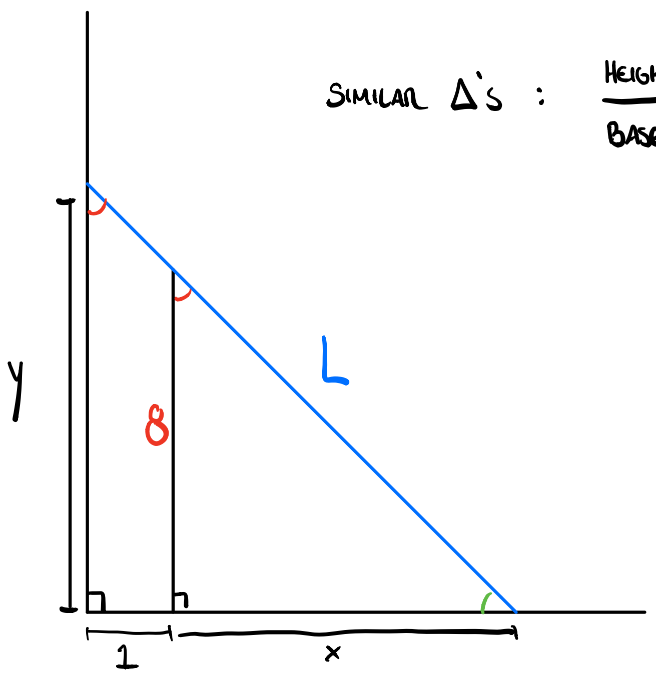
MINIMIZE $L = \sqrt{(x+1)^2 + y^2}$

$x \geq 0 \quad , \quad y \geq 8$

* NOTE: INSTEAD OF MINIMIZING THE LENGTH OF THE LADDER L, LET'S MINIMIZE L^2, THE SQUARE OF THE LENGTH OF THE LADDER. CLEARLY, L^2 IS MINIMIZED WHEN L IS MINIMIZED.

SET $\ell = L^2$.

MINIMIZE $\ell = (x+1)^2 + y^2 \xrightarrow{\text{1 VAR.}}$ $\cancel{(1+x)}\dfrac{y}{\cancel{1+x}} = \dfrac{8}{x}(1+x)$



SIMILAR Δ's : $\dfrac{\text{HEIGHT}}{\text{BASE}} : \underbrace{\dfrac{y}{1+x} = \dfrac{8}{x}}$

$y = \dfrac{8(1+x)}{x} = \dfrac{8}{x} + \dfrac{8x}{x} = \dfrac{8}{x} + 8$

$\ell = (x+1)^2 + \left(\dfrac{8}{x} + 8\right)^2$

$x \in (0, \infty) \leftarrow$ NOT CLOSED INT.

$8x^{-1}$

→ CRITICAL #'s: $\ell' = 2(x+1) + 2\left(\frac{8}{x}+8\right)\left(-\frac{8}{x^2}\right) = 0$

$$2x+2-\frac{16}{x^2}\left(\frac{8}{x}+8\right) = 0$$

$$x^3\left(2x+2-\frac{128}{x^3}-\frac{128}{x^2}\right) = (0)x^3 \quad , \; x>0$$

$$2x^4+2x^3-128-128x = 0$$

$$2(\underbrace{x^4+x^3}\ \underbrace{-64x-64}) = 0$$

$$2\left[x^3(x+1)-64(x+1)\right] = 0$$

$x^3-a^3 = (x-a)(x^2+ax+a^2)$

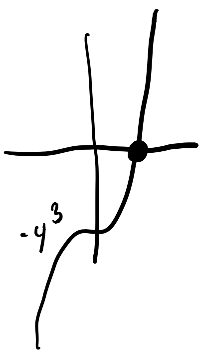
$2(x+1)(x^3-64) = 0$ $\quad$ $2(x+1)(x^3-4^3) = 0$

$$2(x+1)(x-4)(\underbrace{x^2+4x+16}_{\neq 0}) = 0$$

$b^2-4ac = 16-4(16) < 0$

$x^3-4^3=0$

$x=4$

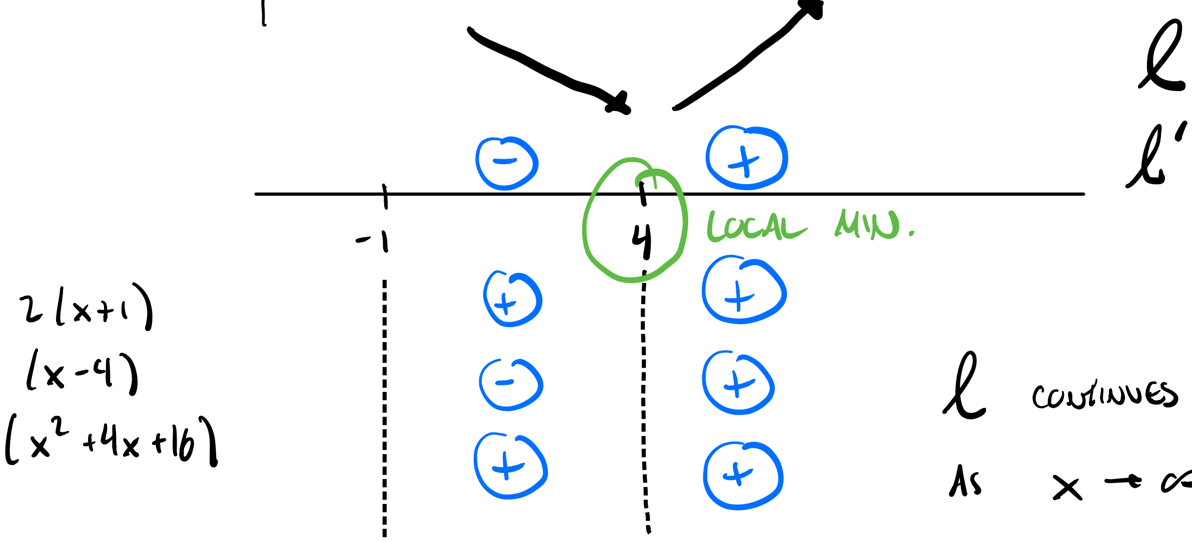


$x=1$

Not in DOMAIN

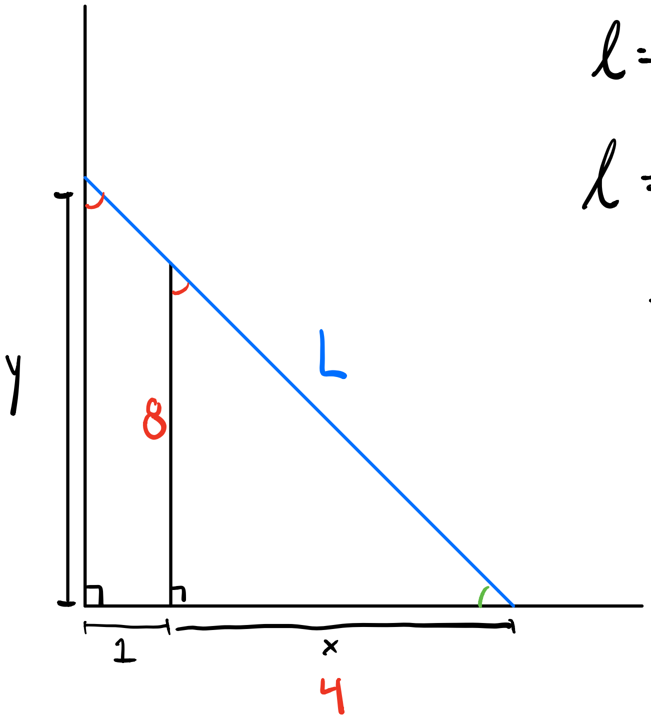
$x=4$

C.P.



ℓ CONTINUES TO INCREAS AS $x \to \infty$,

$x=4$ IS ABS MIN ON $(0,\infty)$



$$\ell = (x+1)^2 + \left(\frac{8}{x} + 8\right)^2$$

$$\ell = (4+1)^2 + \left(\frac{8}{4} + 8\right)^2$$

$$= 5^2 + 10^2 = 125$$

$$L^2 = \ell$$

$$L = \sqrt{\ell} = \sqrt{125} = 5\sqrt{5}$$

63. A retailer has been selling 1200 tablet computers a week at \$350 each. The marketing department estimates that an additional 80 tablets will sell each week for every \$10 that the price is lowered.

(a) Find the demand function.

(b) What should the price be set at in order to maximize revenue?

(c) If the retailer's weekly cost function is

$$C(x) = 35{,}000 + 120x$$

what price should it choose in order to maximize its profit?

Applications to Business and Economics

In Section 2.7 we introduced the idea of marginal cost. Recall that if $C(x)$, the **cost function**, is the cost of producing x units of a certain product, then the **marginal cost** is the rate of change of C with respect to x. In other words, the marginal cost function is the derivative, $C'(x)$, of the cost function.

Now let's consider marketing. Let $p(x)$ be the price per unit that the company can charge if it sells x units. Then p is called the **demand function** (or **price function**) and we would expect it to be a decreasing function of x. (More units sold corresponds to a lower price.) If x units are sold and the price per unit is $p(x)$, then the total revenue is

$$R(x) = \text{quantity} \times \text{price} = xp(x)$$

and R is called the **revenue function**. The derivative R' of the revenue function is called the **marginal revenue function** and is the rate of change of revenue with respect to the number of units sold.

If x units are sold, then the total profit is

$$P(x) = R(x) - C(x)$$

and P is called the **profit function**. The **marginal profit function** is P', the derivative of

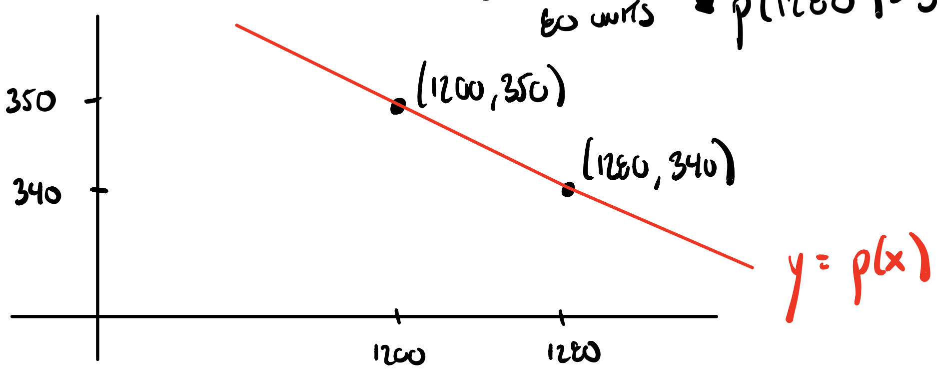
DEMAND FUNCTION: $p(x)$ IS LINEAR,

$p(1200) = 350$

SELL ADDITIONAL 80 UNITS

$p(1280) = 340$

LOWER PRICE BY \$10



$$\text{SLOPE} = \frac{340 - 350}{1280 - 1200} = -\frac{10}{80} = -\frac{1}{8}$$

$$y - 350 = -\frac{1}{8}(x - 1200)$$

$$y = -\frac{1}{8}x + 500$$

$p(x) = 500 - \frac{1}{8}x$

Revenue $R(x) = x\,p(x)$

$R(x) = x\left(500 - \frac{1}{8}x\right)$

Profit $\pi(x) = R(x) - C(x)$

$\pi(x) = x\left(500 - \frac{1}{8}x\right) - \left(35{,}000 + 120x\right)$

Domain: $[0, \infty)$

Critical Points: $\pi'(x) = \left(500 - \frac{1}{8}x\right) + x\left(-\frac{1}{8}\right) - 120$

$0 = 500 - \frac{1}{8}x - \frac{1}{8}x - 120$

$0 = 380 - \frac{1}{4}x$

$x = 4 \cdot 380 = 1440$ # units sold that will maximize profit.

Price $\rightarrow$ $p(1440) = 500 - \frac{1}{8}(1440)$

720

360

180

$= 500 - 180$

$= \$320$