Evaluate the limit by first recognizing the sum as a Riemann sum for a function defined on [0, 4].

$$\lim_{n\to\infty} \frac{4}{n}\left(\sqrt{\frac{4}{n}} + \sqrt{\frac{8}{n}} + \sqrt{\frac{12}{n}} + \cdots + \sqrt{\frac{4n}{n}}\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx$$

DEF. OF RIEMANN INTEGRAL

$$\Delta x = \frac{b-a}{n}$$

$$x_i = a + i\Delta x$$

$a = 0$, $\Delta x = 4/n$, $4 = b$

$0 \quad \frac{4}{n} \quad \frac{8}{n} \quad \frac{12}{n} \quad \cdots \quad \frac{4n}{n}$

$\frac{4(1)}{n} \quad \frac{4(2)}{n} \quad \frac{4(3)}{n} \quad \frac{4n}{n}$

$$\sqrt{\frac{4(1)}{n}} + \sqrt{\frac{4(2)}{n}} + \sqrt{\frac{4(3)}{n}} \cdots + \sqrt{\frac{4n}{n}}$$

$$\sum_{i=1}^{n} \sqrt{\frac{4i}{n}}$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{4i}{n}}\ \frac{4}{n} = \int_0^4 \sqrt{x}\,dx$$

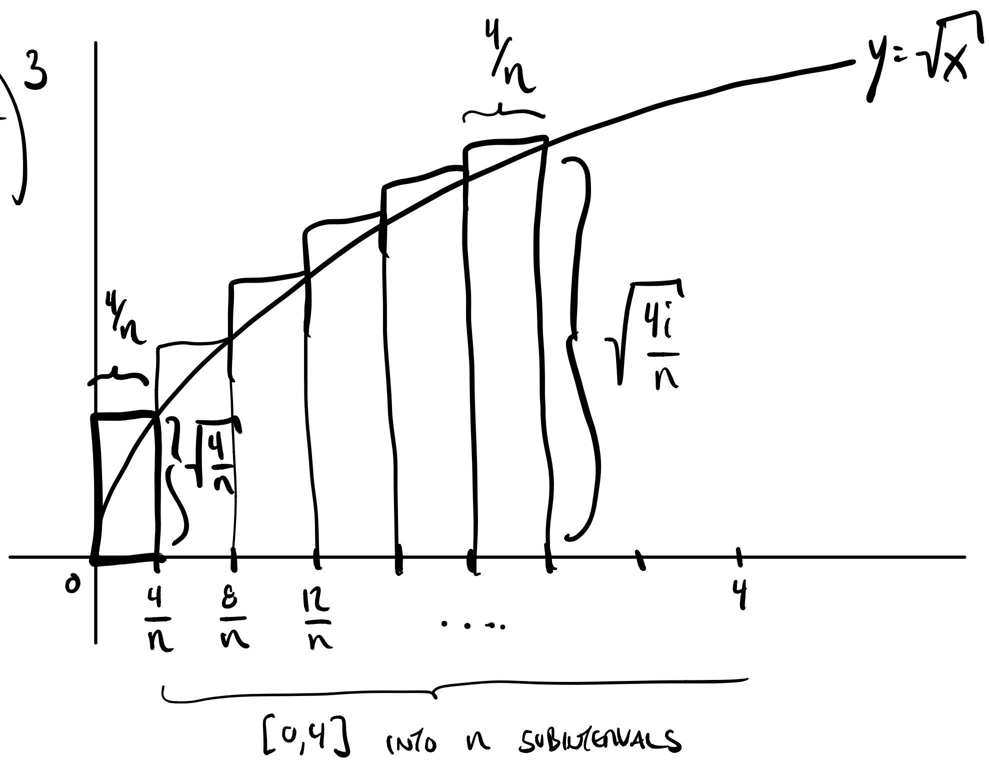
$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

$$x_i = \frac{4i}{n} = 0 + i\,\frac{4}{n}$$

$$f(x) = \sqrt{x} \qquad \Delta x = \frac{4}{n}$$

$$\frac{2}{3}x^{3/2}\Big|_0^4 = \frac{2}{3}\cdot\left(4^{1/2}\right)^3$$

$$= \frac{2}{3}\cdot 2^3 = \frac{16}{3}$$

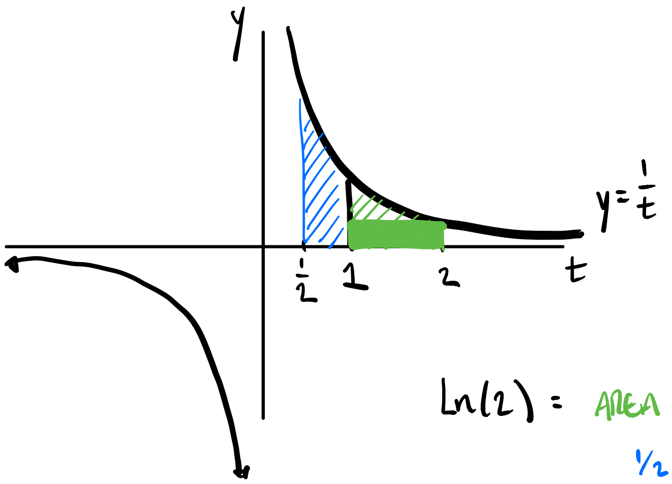


§6.2* The Natural Logarithm

> **1 Definition** The **natural logarithmic function** is the function defined by
> $$\ln x = \int_1^x \frac{1}{t}\,dt \qquad x > 0$$

- Domain
- Pos, Neg, 0
- Increasing (F.T.C.)
- Concave Down
- $\lim_{x \to \infty} \ln x = \infty$

  $\lim_{x \to 0^+} \ln x = -\infty$



$$\ln(1) = \int_1^1 \frac{1}{t}\,dt = 0$$

$$\ln(2) = \text{Area} \ \geq \frac{1}{2}$$

$$\ln\left(\frac{1}{2}\right) = \int_1^{1/2} \frac{1}{t}\,dt = -\int_{1/2}^1 \frac{1}{t}\,dt = -\text{Area}$$

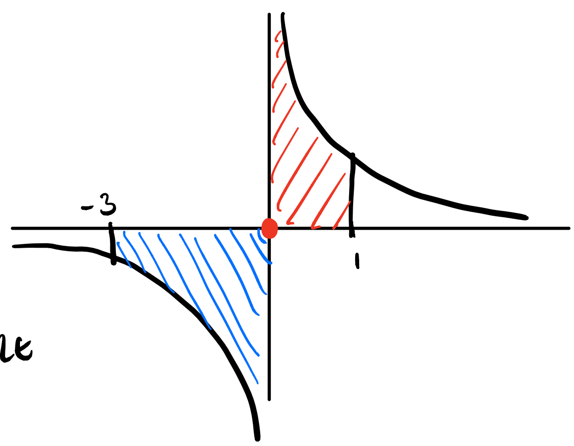
$$\boxed{\text{Domain of } \ln(x) = (0, \infty)}$$

$\int_a^b f(x)\,dx$ defined as long as f is continuous on $[a,b]$.

Could you evaluate

$$\ln(-3) = \int_1^{-3} \frac{1}{t}\,dt \ ?$$

Not defined



Can't integrate across a point where the integrand is undefined.

$$\boxed{\ln(x) \text{ is } \begin{cases} \text{Positive} & \text{if } x > 1 \\ 0 & \text{if } x = 0 \\ \text{Negative} & \text{if } x < 1 \end{cases}}$$

$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \frac{d}{dx}\ln(x) = \frac{1}{x} \qquad \text{(F.T.C.)}$$

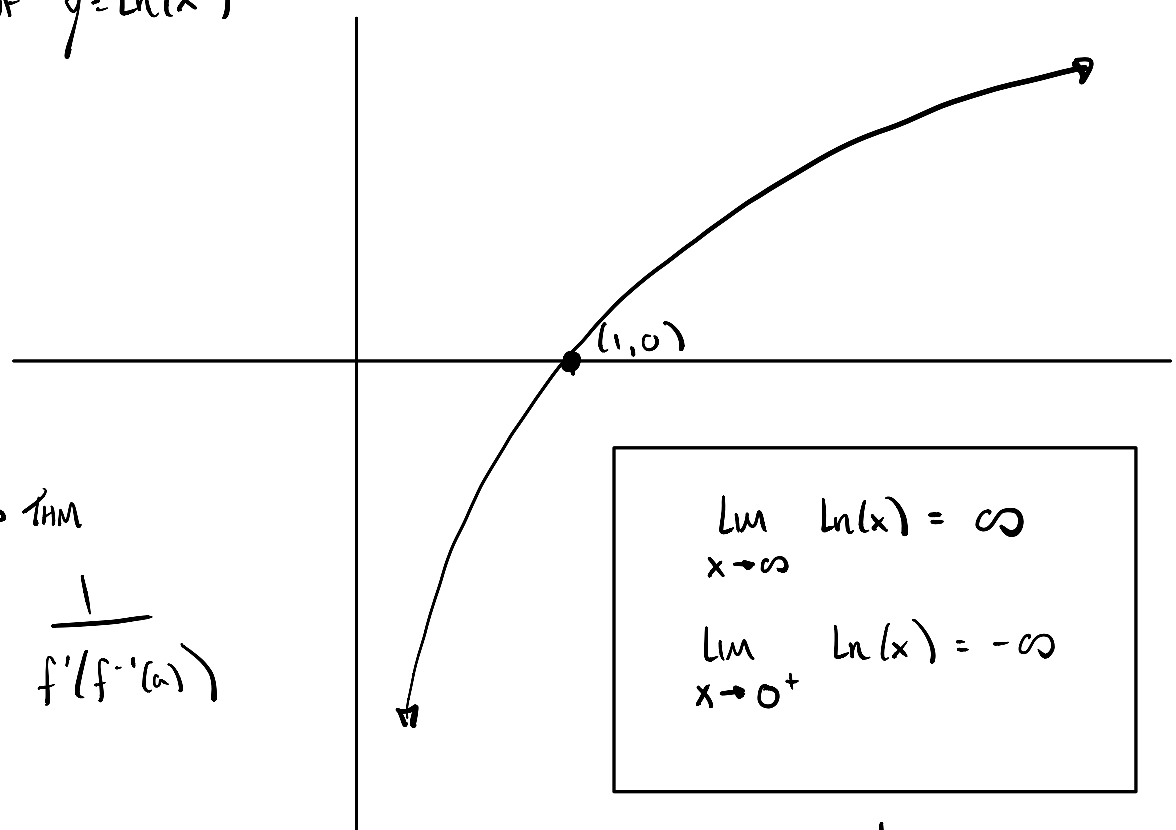
SINCE $x > 0$, $\frac{d}{dx}\ln(x) = \frac{1}{x} > 0$

THAT IS $\boxed{\ln(x) \text{ IS INCREASING ON ITS DOMAIN } (0, \infty)}$

$$\frac{d^2}{dx^2}\ln(x) = \frac{d}{dx}\left[\frac{1}{x}\right] = \frac{-1}{x^2} < 0$$

$\boxed{\text{GRAPH } y = \ln(x) \text{ IS CONCAVE DOWN}}$

GRAPH OF $y = \ln(x)$



$$\boxed{\lim_{x \to \infty} \ln(x) = \infty \qquad \lim_{x \to 0^+} \ln(x) = -\infty}$$

INVERSE FUNCTION THM

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

81. If g is the inverse function of $f(x) = 2x + \ln x$, find $g'(2)$.

$f'(x) = 2 + \frac{1}{x}$, $f^{-1}(2) = 1$

$f(1) = 2$

$$g'(2) = (f^{-1})'(2) \underset{\text{IVT}}{=} \frac{1}{f'(f^{-1}(2))} = \frac{1}{2 + \frac{1}{1}} = \frac{1}{3}$$

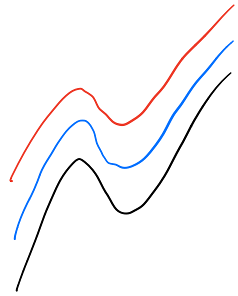
3 Laws of Logarithms If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r\ln x$

1. MULTIPLICATION INSIDE $\longrightarrow$ ADDITION OUTSIDE

2. DIVISION INSIDE $\longrightarrow$ SUBTRACTION OUTSIDE

3. EXPONENTS INSIDE $\longrightarrow$ MULTIPLICATION OUTSIDE



PROOF OF 3: LET $f(x) = \ln(x^r)$

THEN $f'(x) = \frac{1}{x^r} \cdot r x^{r-1} = r \cdot \frac{1}{x} = \frac{d}{dx}\left[r \ln x\right]$

So, $f(x)$ & $r\ln(x)$ HAVE THE SAME DERIVATIVE.

(BOTH ANTIDERIVATIVES OF $r \cdot \frac{1}{x}$)

$\Rightarrow \quad f(x) = r\ln(x) + C$

$\Rightarrow \quad \ln(x^r) = r\ln(x) + C$ FOR ALL x IN DOMAIN

SET $x = 1$: $\ln(1^r) = r\ln(1) + C \Rightarrow 0 = 0 + C \Rightarrow C = 0$

$\therefore \ln(x^r) = r\ln(x)$. $\square$

ex. EXPAND THE LOGARITHMIC EXPRESSION $\ln\left(\dfrac{4a^2b^3}{\sqrt{a^2+b^2}}\right)$

APPLY AS MANY LOG RULES AS POSSIBLE.

3 Laws of Logarithms If x and y are positive numbers and r is a rational number, then

1. $\ln(xy) = \ln x + \ln y$ **2.** $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$ **3.** $\ln(x^r) = r\ln x$

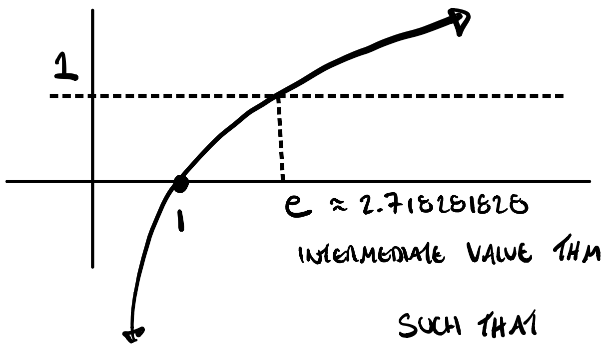
$\overset{(2)}{=} \ln(4a^2b^3) - \ln\left((a^2+b^2)^{1/2}\right)$

$\overset{(1)}{=} \ln(4) + \ln(a^2) + \ln(b^3) - \ln\left((a^2+b^2)^{1/2}\right)$

$\overset{(3)}{=} \boxed{\ln(4) + 2\ln a + 3\ln b - \frac{1}{2}\ln(a^2+b^2)}$

5 Definition e is the number such that $\ln e = 1$.

$\ln(x)$ IS DIFFERENTIABLE $\Rightarrow$ CONTINUOUS



SINCE $\ln(1) = 0$ &

$$\lim_{n\to\infty} \ln(2^n) = \lim_{n\to\infty} n \underbrace{\ln(2)}_{\text{Pos. \#}}$$

$$\geq \lim_{n\to\infty} n \cdot \frac{1}{2} = \infty$$

INTERMEDIATE VALUE THM $\Rightarrow$ THERE IS A # $e \in (1, \infty)$

SUCH THAT $\ln(e) = 1$

$$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx} \qquad \text{or} \qquad \frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)}$$

ex. $\frac{d}{dx} \ln(3x^4 - 4x^3)$ (TWO WAYS)

$$= \frac{1}{3x^4 - 4x^3} \cdot (12x^3 - 12x^2) = \boxed{\frac{12x^3 - 12x^2}{3x^4 - 4x^3}}$$

$$\frac{d}{dx}\left[\ln\left(x^3(3x-4)\right)\right] = \frac{d}{dx}\left[\ln(x^3) + \ln(3x-4)\right]$$

$$= \boxed{\frac{3x^2}{x^3} + \frac{3}{3x-4}}$$

SAME!

ex. $\frac{d}{dx}\left[\ln\left(\frac{x^2-1}{\sqrt{x^2+1}}\right)\right]$ (PRECALC BEFORE CALC)

$$= \frac{d}{dx}\left[\ln(x^2-1) - \frac{1}{2}\ln(x^2+1)\right]$$

$$= \frac{2x}{x^2-1} - \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}x}{x^2+1}$$

$$\frac{d}{dx}(\ln|x|) = \frac{1}{x}$$

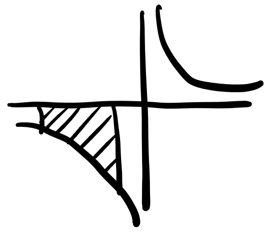
$$\int \frac{1}{x}\,dx = \ln|x| + C$$

CHECK:

$$\frac{d}{dx}\ln|x| = \begin{cases} \frac{d}{dx}\ln x = \frac{1}{x} & \text{IF } x>0 \\ \frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x} & \text{IF } x<0 \end{cases}$$

$$|x| = \begin{cases} x & \text{IF } x \ge 0 \\ -x & \text{IF } x<0 \end{cases}$$

$$\int_{-6}^{-2} \frac{1}{x}\,dx$$



$$F(x)\Big|_{-6}^{-2}$$

$$\left.\begin{matrix} \ln(-6) \\ \ln(-2) \end{matrix}\right\} \text{UNDEFINED}$$

ex. $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

u-SUB: LET $u = \ln x$

$du = \frac{1}{x}\,dx$

$x=e$: $u = \ln(e) = 1$

$x=1$: $u = \ln(1) = 0$

$$\int_{x=1}^{x=e} \xrightarrow{u=\ln x} \int_0^1 u\,du = \frac{1}{2}u^2\Big|_0^1 = \frac{1}{2}\left[(1)^2 - (0)^2\right] = \frac{1}{2}$$

ex. $\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = \int \frac{1}{\cos x} \sin x \, dx$

$u = \cos x$

$du = -\sin x \, dx$

$-du = \sin x \, dx$

$\rightsquigarrow -\int \frac{1}{u} du = -\ln|u| + C$

$\rightsquigarrow -\ln|\cos x| + C$

$= \ln\left(|\cos x|^{-1}\right) + C$

$$\boxed{\int \tan x \, dx = \ln|\sec x| + C}$$

$\cos(x)^{-1} = \frac{1}{\cos x}$

$\cos^{-1}(x) =$ INV OF COS

65–74 Evaluate the integral.

65. $\int_2^4 \frac{3}{x} dx$

66. $\int_0^3 \frac{dx}{5x+1}$

67. $\int_1^2 \frac{dt}{8-3t}$

68. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx$

69. $\int_1^e \frac{x^2+x+1}{x} dx$

70. $\int_e^6 \frac{dx}{x \ln x}$

71. $\int \frac{(\ln x)^2}{x} dx$

72. $\int \frac{\cos x}{2 + \sin x} dx$

73. $\int \frac{\sin 2x}{1 + \cos^2 x} dx$

74. $\int \frac{\cos(\ln t)}{t} dt$

66. $\int_0^3 \frac{1}{5x+1} dx$

LET $u = 5x + 1$

$du = 5 \, dx$

$\frac{1}{5} du = dx$

$\rightsquigarrow \frac{1}{5} \int_1^{16} \frac{1}{u} du = \frac{1}{5} \ln|u| \Big|_1^{16} = \frac{1}{5}(\ln 16 - \ln 1)$

$= \frac{\ln 16}{5}$

Steps in Logarithmic Differentiation

1. Take natural logarithms of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to simplify.
2. Differentiate implicitly with respect to x.
3. Solve the resulting equation for y'.

61–64 Use logarithmic differentiation to find the derivative of the function.

61. $y = (x^2 + 2)^2(x^4 + 4)^4$

62. $y = \dfrac{(x + 1)^4(x - 5)^3}{(x - 3)^8}$

63. $y = \sqrt{\dfrac{x - 1}{x^4 + 1}}$

64. $y = \dfrac{(x^3 + 1)^4 \sin^2 x}{x^{1/3}}$