FINAL EXAM TUE 12/15 1:30 - 3:30 PM ("PAPER")

§6.3* THE NATURAL EXPONENTIAL FUNCTION

$$\ln(x) := \int_1^x \frac{1}{t}\,dt \quad \Rightarrow \quad \frac{d}{dx}\ln(x) = \frac{1}{x} > 0 \quad (x > 0)$$

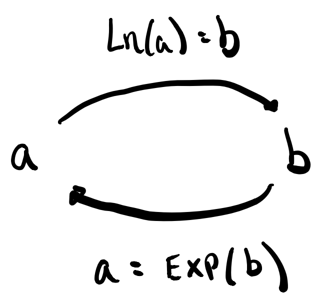
$\text{DOM}(\ln) = (0,\infty)$

$\therefore \ln(x)$ IS INCREASING $\Rightarrow$ $\ln(x)$ IS 1-to-1

$\Downarrow$

$\ln(x)$ HAS AN INVERSE FUNCTION! e^x

LET $\text{EXP}(x) \equiv \ln^{-1}(x)$ TEMP. NAME FOR THE INVERSE OF NATURAL LOGARITHM FUNCTION.



$\text{DOM}(\text{EXP}) = \text{RAN}(\ln) = (-\infty,\infty)$

$\text{RAN}(\text{EXP}) = \text{DOM}(\ln) = (0,\infty)$

$\Downarrow$

$\text{EXP}(x) > 0$

$\text{EXP}(0) = 1 \quad : \quad \ln(1) = 0$

$\text{EXP}(1) = e \quad : \quad \ln(e) = 1$

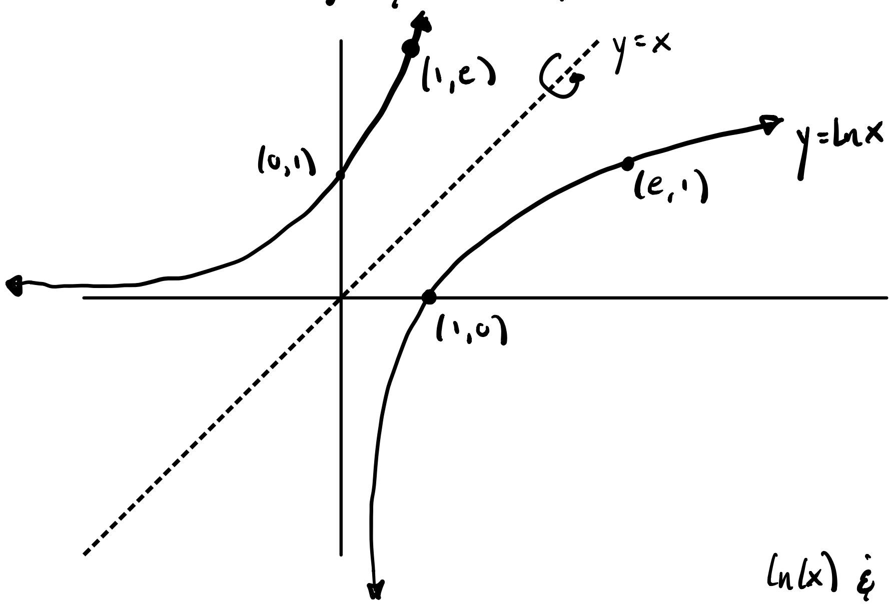
THM: $\ln^{-1}(x) = \text{EXP}(x) = e^x$.

PROOF: $\ln(e^x) = x\ln(e) = x$

$\cancel{\text{EXP}}(\cancel{\ln}(e^x)) = \text{EXP}(x) \quad \Rightarrow \quad \text{EXP}(x) = e^x$

$\square$

NOTE: BOTH $\ln(x)$ & e^x ARE INCREASING FUNCTIONS.



IF $a < b$

THEN $\ln(a) < \ln(b)$ $(a,b>0)$

AND $e^a < e^b$.

$\ln(x)$ & e^x PRESERVE ORDER.

ex. SOLVE THE INEQUALITY $5 < e^{4x-3} \leq 7$

$$\ln(5) < \ln(e^{4x-3}) \leq \ln(7)$$

$$\ln(5) < (4x-3)\underbrace{\ln(e)}_{1} \leq \ln(7)$$

$$\frac{\ln(5)+3}{4} < \cancel{4}x \leq \frac{\ln(7)+3}{4}$$

ex. FIND THE DOMAIN OF $f(x) = \sqrt{10 - e^{3x}}$, & FIND f^{-1}.

$$\text{Dom}(f) = \{x : 10 - e^{3x} \geq 0\} \qquad = (-\infty, \tfrac{1}{3}\ln(10)]$$

$$10 \geq e^{3x}$$

$$\ln(10) \geq \ln(e^{3x}) = 3x$$

$$x \leq \frac{\ln(10)}{3} = \ln(10^{1/3}) = \ln(\sqrt[3]{10})$$

$a < b$

$b > a$

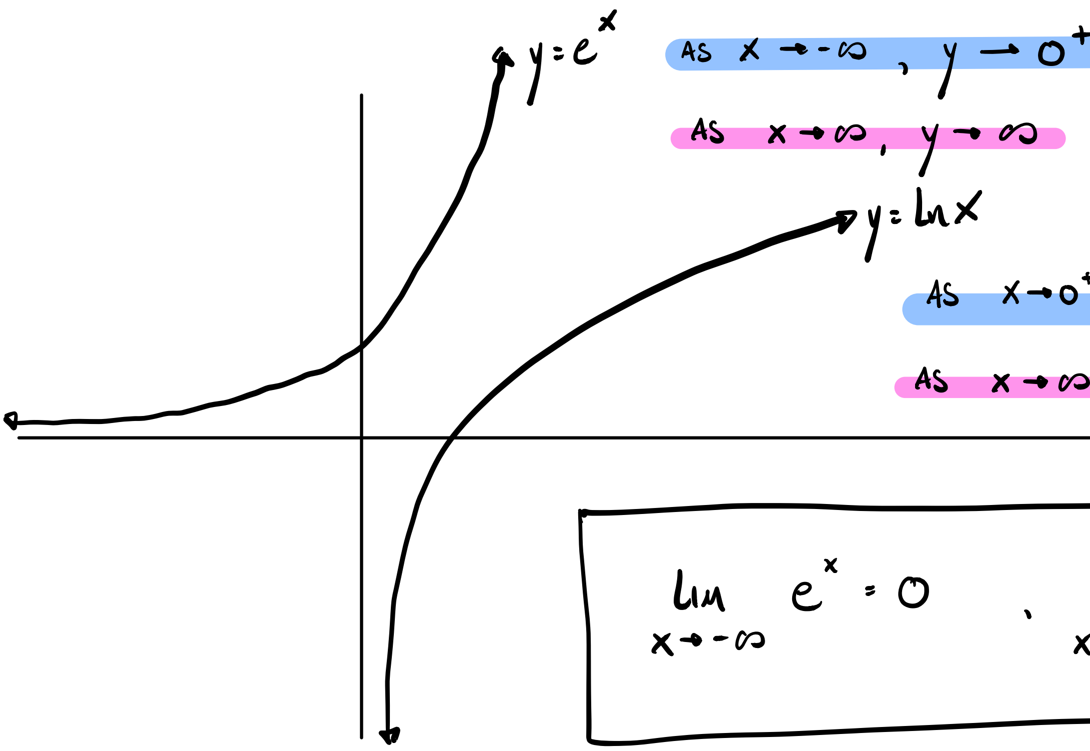
LET $y = f(x)$. SOLVE FOR x. SWITCH $x \leftrightarrow y$. $y = f^{-1}(x)$.

$$y = \sqrt{10 - e^{3x}} \Rightarrow y^2 = 10 - e^{3x}$$

$$e^{3x} = 10 - y^2 \Rightarrow 3x = \ln(10 - y^2)$$

$$x = \frac{1}{3}\ln(10 - y^2)$$

$$f^{-1}(x) = \frac{1}{3}\ln(10 - x^2)$$



AS $x \to -\infty$, $y \to 0^+$

AS $x \to \infty$, $y \to \infty$

AS $x \to 0^+$, $y \to -\infty$

AS $x \to \infty$, $y \to \infty$

$$\boxed{\lim_{x \to -\infty} e^x = 0 \quad , \quad \lim_{x \to \infty} e^x = \infty}$$

ex. FIND THE LIMIT $\displaystyle\lim_{x \to \infty} \frac{e^{x^2} - e^{-x^2}}{e^{x^2} + e^{-x^2}}$

IDEA

$$\left(\approx \frac{e^{x^2} - 0}{e^{x^2} + 0} = 1 \, ? \right)$$

$$= \lim_{x \to \infty} \frac{\cancel{e^{x^2}}(1 - e^{-2x^2})}{\cancel{e^{x^2}}(1 + e^{-2x^2})}$$

$$= \frac{\lim_{x \to \infty} (1 - e^{-2x^2})}{\lim_{x \to \infty} (1 + e^{-2x^2})}$$

$$\left(\lim_{x \to \infty} \frac{3x^3 + 1}{4x^3 - 2x} \right.$$

$$\left. \lim_{x \to \infty} \frac{\cancel{x^3}(3 + \frac{1}{x^3})}{\cancel{x^3}(4 - \frac{2}{x^2})} = \frac{3}{4} \right)$$

$$= \frac{1-0}{1+0} = \boxed{1}$$

$$e^{(x^y)}$$

$$(e^x)^y \neq e^{x^y}$$

Laws of Exponents:

1. $e^x e^y = e^{x+y}$
2. $\frac{e^x}{e^y} = e^{x-y}$
3. $(e^x)^y = e^{xy}$

Proof of 3: $\ln\left((e^x)^y \right) = y \cancel{\ln} (\cancel{e}^x)$ Log rule 3

$= yx$

$$\cancel{e}^{\cancel{\ln}\left((e^x)^y \right)} = e^{yx}$$

$$(e^x)^y = e^{xy}$$

$\square$

Calculus

$$\boxed{\frac{d}{dx} e^x = e^x}$$

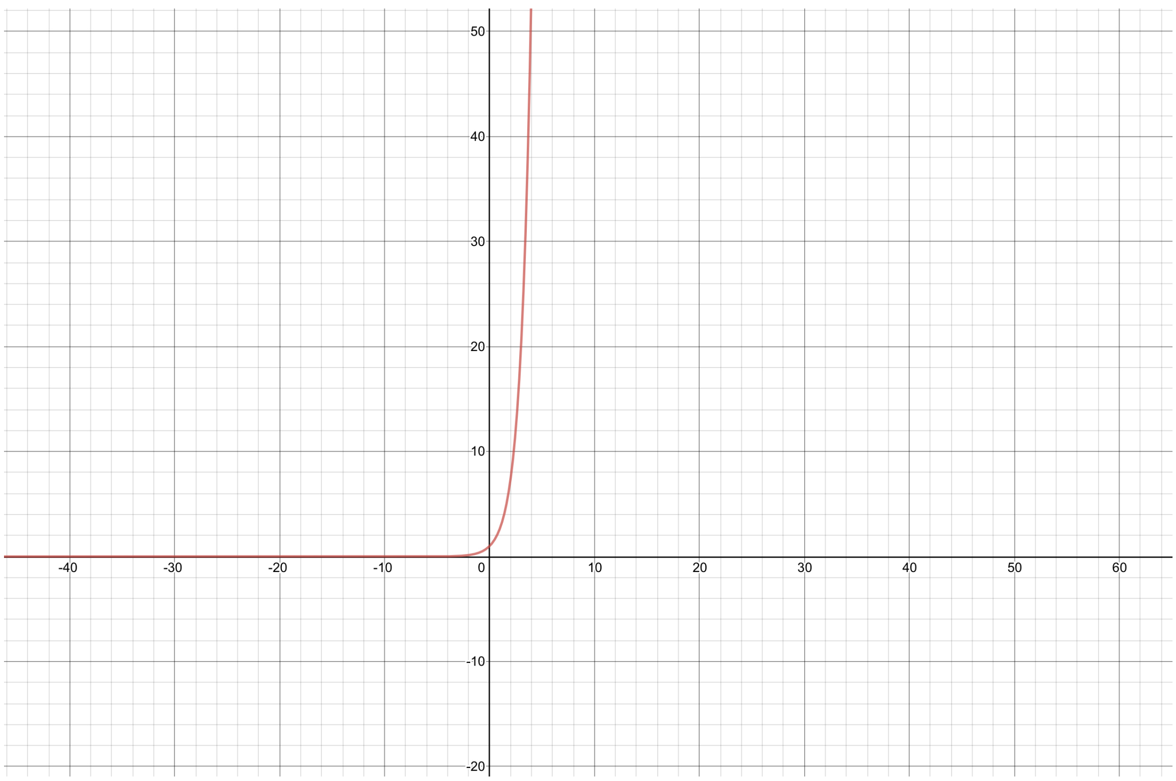
Proof: Let $y = e^x$. Find y'. Explicit

$\ln y = x$ Implicit

$$\frac{d}{dx} \ln y = \frac{d}{dx} x \Rightarrow \frac{d}{dy} \ln y \frac{dy}{dx} = \frac{d}{dx} x$$

$$\frac{1}{y}\, y' = 1 \quad \Rightarrow \quad y' = y$$

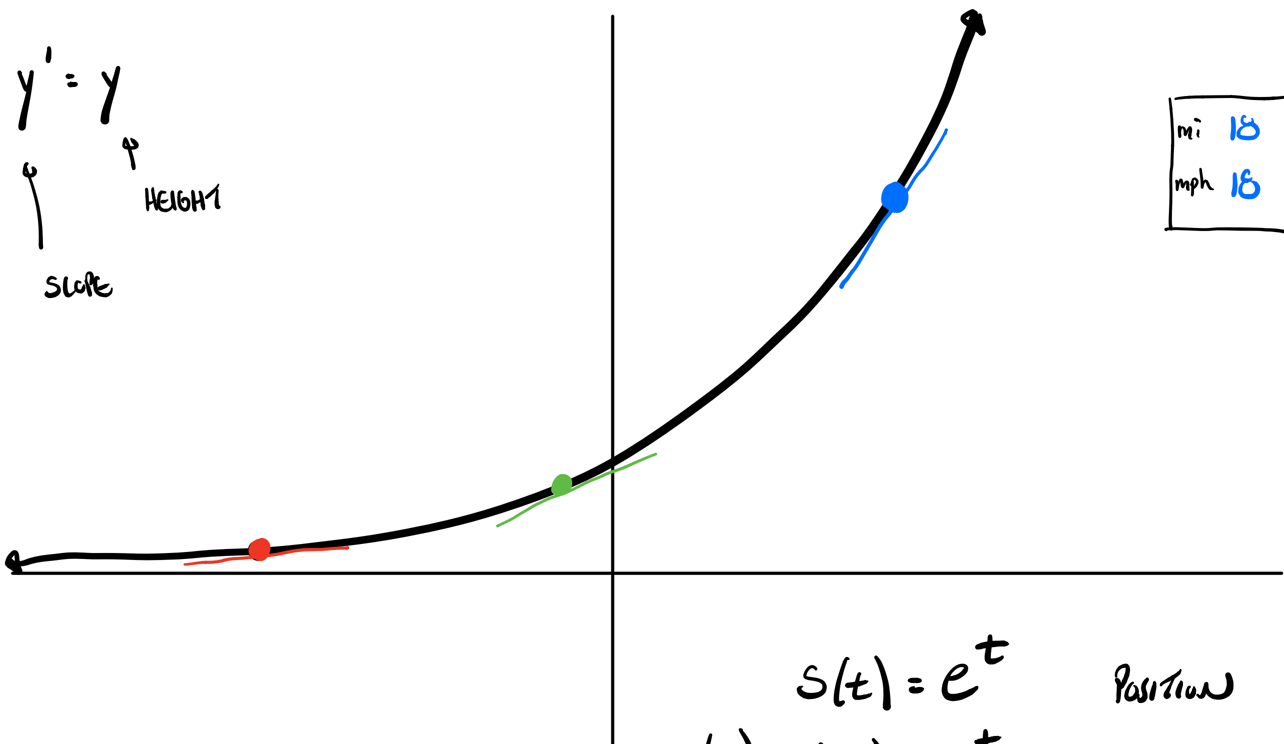
$$y' = e^x \qquad \square$$



$y' = y$

SLOPE

HEIGHT



mi	18
mph	18

$s(t) = e^t$ POSITION ODOMETER

$v(t) = s'(t) = e^t$ VELOCITY SPEEDOMETER

$\underline{\text{ex}}$. DIFFERENTIATE $f(x) = x^2 e^{-1/x}$

$$f'(x) = 2x e^{-1/x} + x^2 \frac{d}{dx}\left[e^{-1/x}\right]$$

LET $u = -1/x = -x^{-1}$

$$\frac{du}{dx} = x^{-2}$$

$$\frac{d}{dx}\left[e^u\right] = \frac{d}{du}\left[e^u\right]\frac{du}{dx} = e^u x^{-2}$$

$$f'(x) = 2x e^{-1/x} + \cancel{x^2} e^{-1/x} \cancel{x^{-2}}.$$

59. For what values of r does the function $y = e^{rx}$ satisfy the differential equation $\underbrace{y'' + 6y' + 8y = 0}_{\text{DIFF. EQ.}}$?

$y = e^{rx}$

$y' = e^{rx} \frac{d}{dx}[rx] = re^{rx}$

$y'' = r^2 e^{rx}$

$r^2 e^{rx} + 6re^{rx} + 8e^{rx} = 0$ SOLVE FOR r

$e^{rx}(r^2 + 6r + 8) = 0$

$e^{rx}(r+4)(r+2) = 0$

$e^{rx} \neq 0$

$$\boxed{r = -4 \quad \text{OR} \quad r = -2}$$

56. Find an equation of the tangent line to the curve $xe^y + ye^x = 1$ at the point $(0, 1)$.

$$y - 1 = m(x - 0)$$

$$m = \left.\frac{dy}{dx}\right|_{(0,1)}$$

IMPLICIT DIFF.

$$\frac{d}{dx}\left[xe^y + ye^x\right] = \frac{d}{dx}[1]$$

$$1e^y + xe^y\frac{dy}{dx} + \frac{dy}{dx}e^x + ye^x = 0$$

$$xe^y\frac{dy}{dx} + e^x\frac{dy}{dx} = -ye^x - e^y$$

$$\frac{dy}{dx}\left(xe^y + e^x\right) = -ye^x - e^y$$

$$\frac{dy}{dx} = \frac{-ye^x - e^y}{xe^y + e^x} \longrightarrow \left.\frac{dy}{dx}\right|_{(0,1)} = \frac{-e^0 - e^1}{e^0}$$

$$= -1 - e$$

$$\boxed{y - 1 = (-1 - e)\,x}$$