FINAL EXAM TUE 12/15 1:30 - 3:30 PM (ZOOM PDF / "PAPER")

Fordham Math 1206, Calculus I **Final Exam Practice Problems**

1. Find the limit or state that it does not exist. Justify your answer.

   (a) $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-x}\right)$

   (b) $\lim_{x\to 0^+}\left(\ln(1/x)-\ln(1/\sin 4x)\right)$

   (c) $\lim_{x\to\infty}\dfrac{\sin x}{\ln x}$

STRAIGHTFORWARD LIMIT / DERIVATIVE / INTEGRAL

$$(A-B)(A+B) = A^2 - B^2$$

(a) $\lim_{x\to\infty}\left(\sqrt{x^2+2x}-\sqrt{x^2-x}\right)\dfrac{\sqrt{x^2+2x}+\sqrt{x^2-x}}{\sqrt{x^2+2x}+\sqrt{x^2-x}}$

$$= \lim_{x\to\infty}\frac{(x^2+2x)-(x^2-x)}{\sqrt{x^2+2x}+\sqrt{x^2-x}} = \lim_{x\to\infty}\frac{3x}{\sqrt{x^2+2x}+\sqrt{x^2-x}}$$

$$\left[\begin{aligned}\sqrt{x^2+2x}+\sqrt{x^2-x} &= \sqrt{x^2\left(1+\tfrac{2}{x}\right)}+\sqrt{x^2\left(1-\tfrac{1}{x}\right)}\\ &= \sqrt{x^2}\sqrt{1+\tfrac{2}{x}}+\sqrt{x^2}\sqrt{1-\tfrac{1}{x}}\\ &= x\left(\sqrt{1+\tfrac{2}{x}}+\sqrt{1-\tfrac{1}{x}}\right)\end{aligned}\right]$$

$$= \lim_{x\to\infty}\frac{3\cancel{x}}{\cancel{x}\left(\sqrt{1+\frac{2}{x}}+\sqrt{1-\frac{1}{x}}\right)}$$

$$\frac{\lim_{x\to\infty} 3}{\lim_{x\to\infty}\left(\sqrt{1+\underbrace{\tfrac{2}{x}}_{0}}+\sqrt{1-\underbrace{\tfrac{1}{x}}_{0}}\right)} = \frac{3}{2}$$

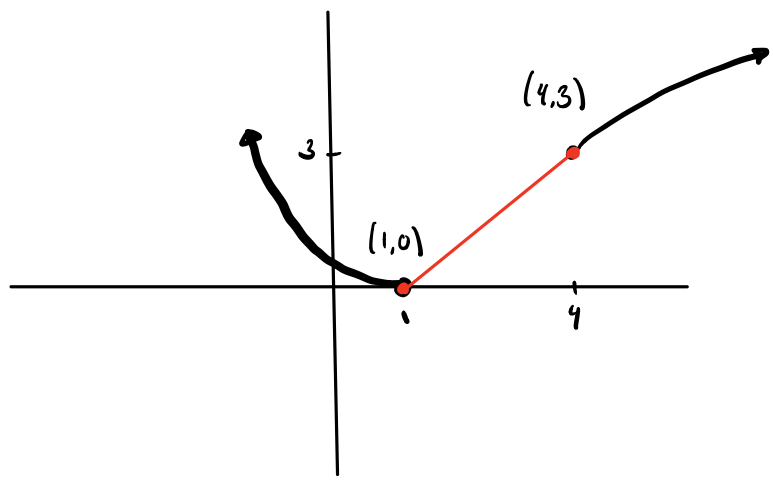
2. Find a and b such that the following function is continuous on $\mathbb{R}$.

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x < 1 \\ ax + b & \text{if } 1 \leq x \leq 4 \\ \sqrt{2x+1} & \text{if } x > 4 \end{cases}$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x) = f(1)$$

$$\lim_{x \to 4^-} f(x) = \lim_{x \to 4^+} f(x) = f(4)$$

$$f(x) = \begin{cases} (x-1)^2 & \text{if } x < 1 \\ ax + b & \text{if } 1 \leq x \leq 4 \\ \sqrt{2x+1} & \text{if } x > 4 \end{cases}$$



$2x + 1 \geq 0$

$2x \geq -1$

$x \geq -\frac{1}{2}$

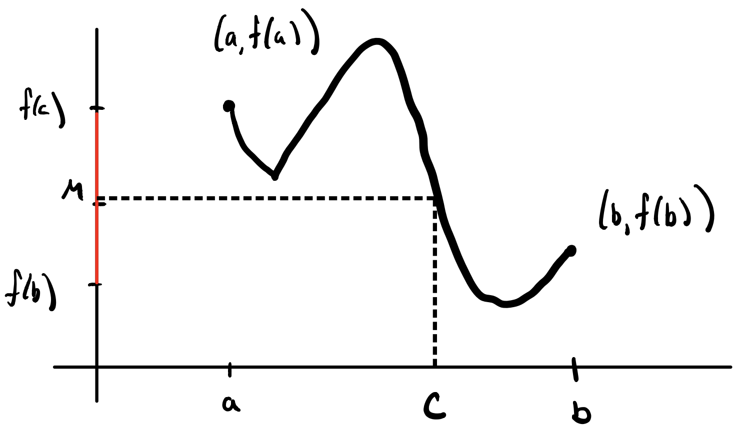
3. (a) State the Intermediate Value Theorem, including the hypotheses.
   (b) Prove that the following equation has at least one real solution.

$$\sqrt{x-5} = \frac{1}{x+3}$$

IF f IS CONT. ON $[a, b]$ AND IF M IS

A NUMBER BETWEEN $f(a)$ & $f(b)$

THE THERE EXISTS SOME $a < c < b$ SUCH THAT $f(c) = M$.



$\sqrt{x-5} = \frac{1}{x+3}$ $\longrightarrow$ $\sqrt{x-5} - \frac{1}{x+3} = 0$

SHOW $\exists$ A NUMBER c S.T. $f(c) = 0$

$$f(x) = \sqrt{x-5} - \frac{1}{x+3}$$

$f(5) = -\frac{1}{8}$, $f(9) = 2 - \frac{1}{12} = \frac{23}{12}$ $\qquad -\frac{1}{8} < 0 < \frac{23}{12}$

SO $\exists$ $5 < c < 9$ S.T. $f(c) = 0$.

or $f'(a)$

4. Let $f(x) = \dfrac{2}{3x}$. Use the definition of the derivative as a limit to find $f'(x)$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

5. Find $\frac{dy}{dx}$.

(a) $y = \sin(x^2)\cos^4(\sqrt{x})$

(b) $y = \frac{(2x^5-1)^4(3x^4+5)^3}{\sqrt{x^2+1}}$

(c) $x^y = y^x$

(d) $y = \int_0^{\tan x} \sqrt{1-t^3}\,dt$ (FUND. THM. OF CALC.)

PRODUCT RULE

QUOTIENT RULE

POWER RULE

LOG / EXP.

IMPLICIT DIFF.

* LOGARITHMIC DIFF.

$$\sqrt{1-(\tan x)^3} \quad \sec^2 x$$

CHAIN RULE : $\frac{d}{dx}[\tan x]$

(b) $y = \frac{(2x^5-1)^4(3x^4+5)^3}{\sqrt{x^2+1}}$ $\xrightarrow{*}$ $\ln y = 4\ln|2x^5-1| + 3\ln(3x^4+5) - \frac{1}{2}\ln(x^2+1)$

$$\frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}\left[4\ln|2x^5-1| + 3\ln(3x^4+5) - \frac{1}{2}\ln(x^2+1)\right]$$

$$\frac{dy}{dx} = y\frac{d}{dx}\left[4\ln|2x^5-1| + 3\ln(3x^4+5) - \frac{1}{2}\ln(x^2+1)\right]$$

$$= \frac{(2x^5-1)^4(3x^4+5)^3}{\sqrt{x^2+1}}\frac{d}{dx}\left[4\ln|2x^5-1| + 3\ln(3x^4+5) - \frac{1}{2}\ln(x^2+1)\right]$$

$$= \frac{(2x^5-1)^4(3x^4+5)^3}{\sqrt{x^2+1}}\left[\frac{4\cdot 10x^4}{2x^5-1} + \frac{3\cdot 12x^3}{3x^4+5} - \frac{2x}{2(x^2+1)}\right]$$

6. Find the tangent line to the curve $y = \sin(\pi x)e^{-}x$ at the point $(2, 0)$.

$$y = \sin(\pi x)e^{-x}$$

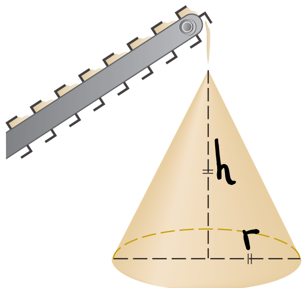
EQ OF TANGENT LINE TO $y = f(x)$ AT $(a, f(a))$ IS

$$y - f(a) = f'(a)(x - a)$$

$$\boxed{y = f(a) + f'(a)(x - a)}$$

$$\frac{dV}{dt}$$

7. Gravel is being dumped from a conveyer belt at a rate 30 cubic feet per minute, and its courseness is such that it forms a pile in the shape of a cone whose base diameter and height are equal. How fast is the height of the pile increasing when the pile is 10 feet tall?



QUANTITIES : $V = \frac{\pi}{12} h^3$

$\Big\downarrow \frac{d}{dt}$

RATES $\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$

$$\left.\begin{array}{l} h = 2r \rightarrow r = \frac{h}{2} \\ V = \frac{1}{3}\pi r^2 h \end{array}\right\} \quad V = \frac{\pi}{3}\left(\frac{h}{2}\right)^2 h$$

$$\frac{d}{dt}\left[V\right] = \frac{d}{dt}\left[\frac{\pi}{12} h^3\right] = \frac{d}{dh}\left[\frac{\pi}{12} h^3\right]\frac{dh}{dt}$$ CHAIN RULE

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt}$$

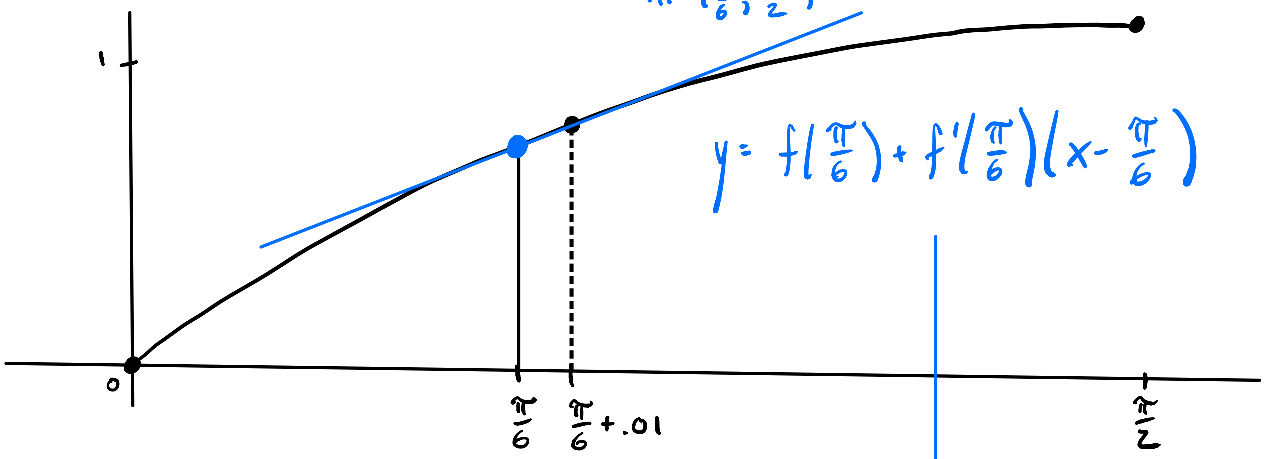
$$\frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt} \longrightarrow \frac{dh}{dt} = \frac{4}{\pi (10)^2}(30)$$

$$= \frac{4}{\pi\, 100\, ft^2}\left(30\, ft^3/\text{min}\right)$$

$$= \frac{120}{100\pi}\ ft/\text{min}$$

8. Use differentials/linear approximation to estimate $\sin(\pi/6 + 0.01)$. You do not need to simplify your answer.

TANGENT LINE TO $y = \sin(x)$ AT $\left(\frac{\pi}{6}, \frac{1}{2}\right)$



$$y = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x - \frac{\pi}{6}\right)$$

$f(x) = \sin x$

$f'(x) = \cos(x) \qquad f'\left(\frac{\pi}{6}\right) = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

$$y = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) = L(x)$$

WHEN x IS CLOSE TO $\frac{\pi}{6}$,

$f(x) \approx L(x)$

$$f\left(\frac{\pi}{6} + .01\right) \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{6} + .01 - \frac{\pi}{6}\right) = \frac{1}{2} + \frac{\sqrt{3}}{2}(.01)$$

DOMAIN $[0,\infty)$

9. (a) State the Extreme Value Theorem, including the hypotheses.

   (b) Find the absolute maximum and absolute minimum value of $f(x) = \dfrac{\sqrt{x}}{1+x^2}$ over the interval $[0, 2]$

IF f IS CONT. ON $[a,b]$ THEN f ATTAINS AN ABS MAX & ABS MIN VALUE ON $[a,b]$.

ABS. EXTREME VALUE MUST OCCUR AT EITHER

(1) CRITICAL POINTS OR

(2) ENDPOINTS

10. (a) State the Mean Value Theorem, including the hypotheses.

(b) Suppose f(x) is a differentiable function such that $f(1) = 2$ and $f'(x) \geq 5$ for all x. What is the smallest possible value for $f(4)$? Justify your answer.

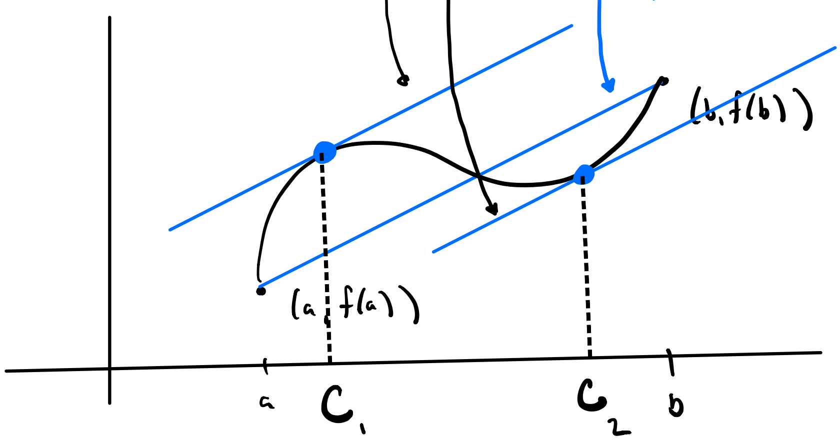
IF f IS CONT. ON $[a, b]$ & DIFFERENTIABLE ON (a, b)

THEN $\exists$ $a < c < b$ S.T.

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

SLOPE

SLOPE



11. Consider the function

$$f(x) = \frac{x^2}{x^2-1}.$$

(a) Find the domain of f.

(b) Find the intervals on which f is increasing/decreasing.

(c) Find the intervals on which f is concave up/down.

(d) Sketch the graph $y = f(x)$. Label all intercepts, asymptotes, local maxima/minima, and inflection points.

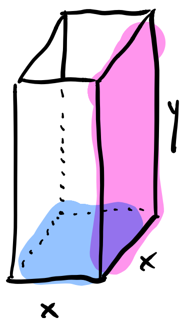
VERICAL $\lim_{x \to a^{+/-}} f(x) = \pm\infty$?

HORIZ. $\lim_{x \to \pm\infty} f(x) = L$?

12. A rectangular box has a square base and an open top. The four sides are made of wood that costs $2 per square foot, while the base is made of aluminum that costs $25 per square foot. If the volume of the box is to be 50 cubic feet, what is its minimum possible cost?

MINIMIZE COST SUBJECT TO CONSTRAINTS.

$$V = x^2 y = 50 \qquad x \geq 0 \quad y \geq 0$$



$$C = 25\,(\text{SQ. FT. OF AL.}) + 2\,(\text{SQ. FT. OF WOOD})$$

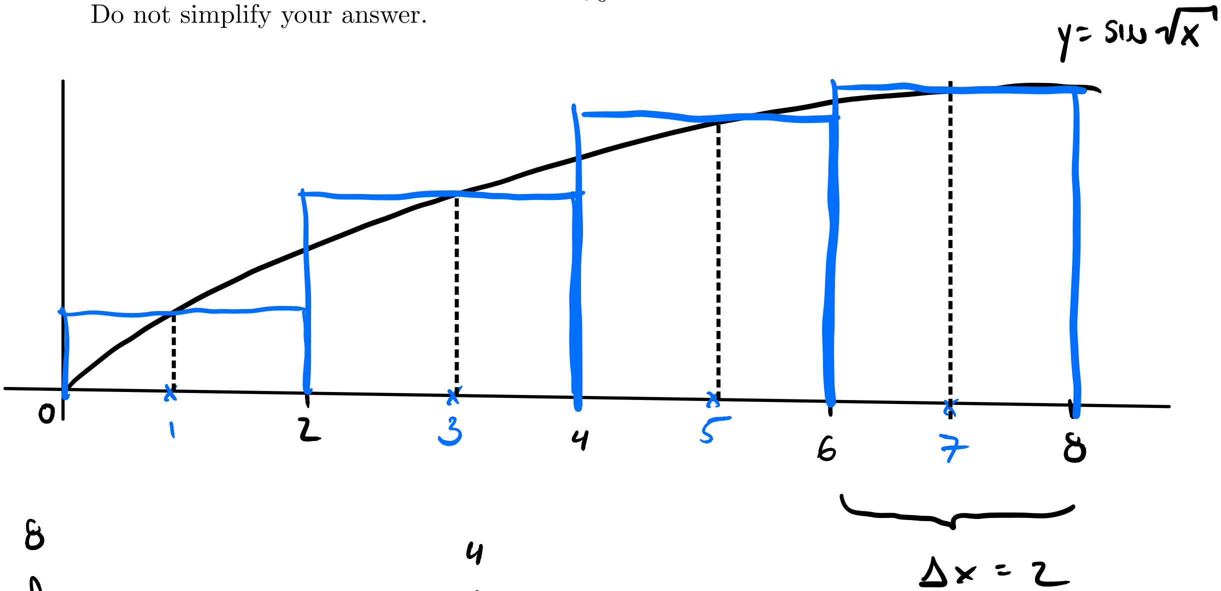
$$C = 25x^2 + 2 \cdot 4xy$$

$\vdots$

or LEFT ENDPOINTS

or RIGHT ENDPOINTS

13. Write down a Riemann sum that estimates $\int_0^8 \sin\sqrt{x}\,dx$ using the midpoint rule with $n = 4$ subdivisions. Do not simplify your answer.



$$\int_0^8 \underbrace{\sin\sqrt{x}}_{f(x)}\,dx \approx \sum_{i=1}^{4} f(\bar{x}_i)\,\Delta x$$

$$\sum_{i=1}^{4} \sin\sqrt{\bar{x}_i}\,(2)$$

$$= \boxed{\left(\sin\sqrt{1} + \sin\sqrt{3} + \sin\sqrt{5} + \sin\sqrt{7}\right)(2)}$$

14. Evaluate the integral.

(a) $\displaystyle\int_1^e \frac{\ln(x^6)}{x}\,dx$

(b) $\displaystyle\int \sin^3(3x)\cos(3x)\,dx$

(c) $\displaystyle\int \frac{36}{(2x+1)^3}\,dx$

(d) $\displaystyle\int x^2 2^{x^3}\,dx$

(a) $\displaystyle\int_1^e \frac{\ln(x^6)}{x}\,dx = 6\int_1^e \frac{\ln x}{x}\,dx$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

...

(d) $\displaystyle\int x^2 2^{x^3}\,dx$

let $u = x^3$

$du = 3x^2\,dx \longrightarrow \frac{1}{3}\,du = x^2\,dx$

$$\int 2^{x^3} \cdot x^2\,dx \rightsquigarrow \frac{1}{3}\int 2^u\,du$$

$$= \frac{1}{3}\cdot\frac{1}{\ln 2}\,2^u + C$$

$$\rightsquigarrow \frac{1}{3\ln 2}\,2^{x^3} + C$$