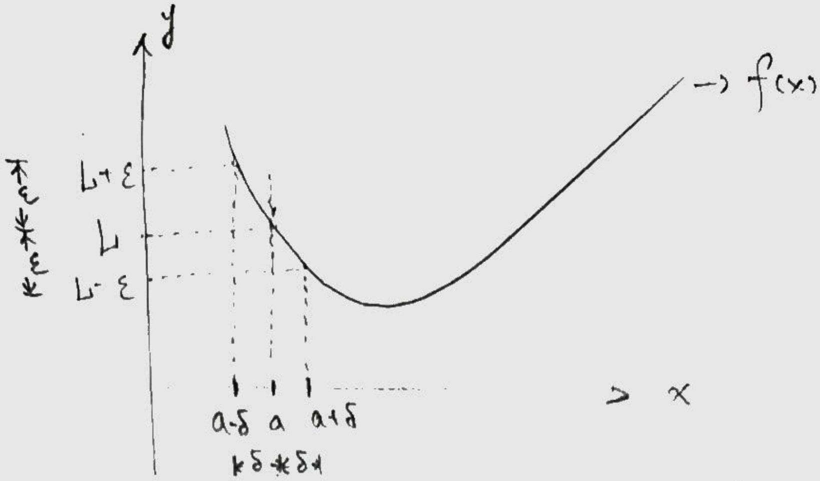
§1.7 The Precise Definition of Limit



① If $x \in (a-\delta, a+\delta)$, then $|x-a| < \delta$

② Similarly, for $x \in (a-\delta, a+\delta)$, the graph above shows that $|f(x)-L| < \varepsilon$

Q: What happens if ε is getting smaller ?

A: We can always find a $\delta > 0$ such that $|f(x)-L| < \varepsilon$.

<u>Precise Def. of Limit</u>. $\lim\limits_{x \to a} f(x) = L$ means that

If for every $\varepsilon > 0$, there exists a number $\delta > 0$, such that

$$\text{if } 0 < |x-a| < \delta, \text{ then } |f(x)-L| < \varepsilon.$$

Example 1. Show that $\lim\limits_{x \to 3} (\overbrace{4x-5}^{f(x)}) = \underbrace{7}_{L}$

Goal: find that δ such that if $0<|x-3|<\delta$, then $|(4x-5)-7| < \varepsilon$

Derivation : $|(4x-5)-7| < \varepsilon$

IDEA: Choose this as δ

$\Rightarrow |4x-12| < \varepsilon \Rightarrow 4|x-3| < \varepsilon \Rightarrow |x-\underbrace{3}_{a}| < \dfrac{\varepsilon}{4}$

PROOF: Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{4} > 0$

If $0 < |x-3| < \delta$, then $|(4x-5)-7| < \varepsilon$.

Therefore, by the precise definition of limit, $\lim_{x \to 3} (4x-5) = 7$.

Example 2. Show that $\lim_{x \to 3} x^2 = 9$

Derivation: $|x^2 - 9| < \varepsilon \Rightarrow |(x+3)(x-3)| < \varepsilon$

$$\Rightarrow |x+3||x-3| < \varepsilon$$

If $|x+3|$ can be bounded by a number C, then

$|x+3| < C$ and $|x-3| < \frac{\varepsilon}{C}$.

Suppose that $|x-3| < \boxed{1}$ then $-1 < x-3 < 1$

$$\Rightarrow 2 < x < 4$$

$$\Rightarrow 5 < x+3 < 7$$

$$\Rightarrow |x+3| < 7 \quad C.$$

Hence, we have $|x-3| < \boxed{\frac{\varepsilon}{7}}$

PROOF: Given $\varepsilon > 0$, let $\delta = \min\{1, \frac{\varepsilon}{7}\}$.

If $0 < |x-3| < \delta$, then $|x^2 - 9| < \varepsilon$.

Therefore, by the definition, $\lim_{x \to 3} x^2 = 9$.

PROOF of the Limit Laws (Sum)

We already know $\lim_{x \to a} (f(x) + g(x)) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = L + M$

if $\lim_{x \to a} f(x) = L$ & $\lim_{x \to a} g(x) = M$ and both exist.

Let's use the precise def to prove it!

IDEA: Given $\varepsilon > 0$, can we find a $\delta > 0$ s.t.

if $0 < |x-a| < \delta$, then $|f(x) + g(x) - (L+M)| < \varepsilon$?

(Triangle inequality. $|a+b| \le |a| + |b|$)

We know that $|f(x) + g(x) - (L+M)| \le \underbrace{|f(x) - L|}_{\varepsilon/2} + \underbrace{|g(x) - M|}_{\varepsilon/2} = \boxed{\varepsilon}$

PROOF: Given $\varepsilon/2 > 0$, there exists $\delta_1 > 0$ such that

if $0 < |x-a| < \delta_1$, then $|f(x) - L| < \varepsilon/2$.

Similarly, there exists $\delta_2 > 0$, such that

if $0 < |x-a| < \delta_2$, then $|g(x) - M| < \varepsilon/2$

Let $\delta = \min\{\delta_1, \delta_2\}$. If $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon/2$ and

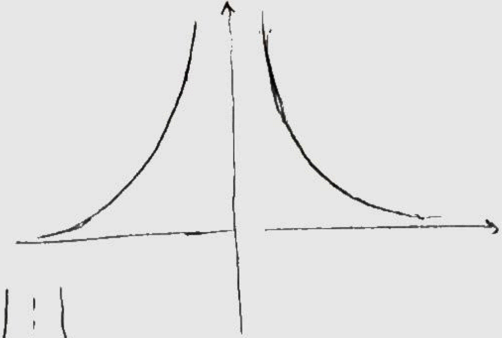
$|g(x) - M| < \varepsilon/2$ and.

Therefore, $|f(x) + g(x) - (L+M)| \le |f(x) - L| + |g(x) - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Thus, by the definition of limit,

$$\lim_{x \to a} (f(x) + g(x)) = L + M \quad \square$$

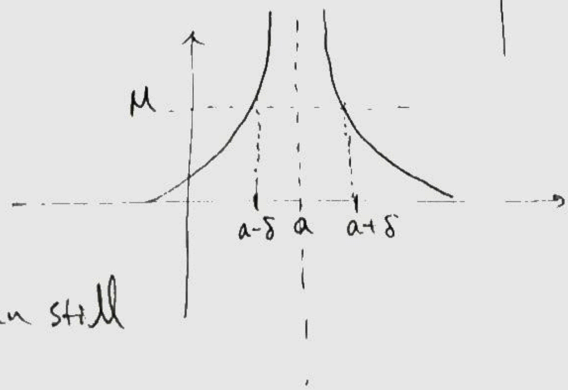
Infinite Limit. $\lim_{x \to a} f(x) = +\infty$



for example $f(x) = \frac{1}{x^2}$, $\lim_{x \to 0} f(x) = +\infty$

① If $x \in (a-\delta, a+\delta)$

then $f(x) > M$.



② If M is getting larger, we can still find $\delta > 0$

Precise Def of Inf. Limit.

For every $M > 0$, there exists a number $\delta > 0$ such that

If $0 < |x-a| < \delta$, then $f(x) > M$.

Example 3. Show that $\lim_{x \to 0} \frac{1}{x^2} = +\infty$.

Goal: Choose a right δ.

Scratch: $|x-0| < \delta \Rightarrow |x| < \delta$.

$\frac{1}{x^2} > M \Rightarrow \frac{1}{|x|} > \sqrt{M}$

$\Rightarrow |x| < \boxed{\frac{1}{\sqrt{M}}} \to \delta$

PROOF: For every $M > 0$, Choose $\delta = \frac{1}{\sqrt{M}}$

If $0 < |x-0| < \delta$, then $\frac{1}{x^2} > M$

This shows that $\frac{1}{x^2} \to +\infty$ as $x \to 0$.

(self-reading: PROOF of $\lim_{x \to a} f(x) = -\infty$)