Let $V(r) = \frac{4}{3}\pi r^3$ Volume of ball is a function of radius.

Goal:

$$\lim_{\Delta r \to 0} \frac{\Delta V}{\Delta r} = V'(2) \underset{\text{by def}}{=} \lim_{h \to 0} \frac{V(2+h) - V(2)}{h}.$$

$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi(2+h)^3 - \frac{4}{3}\pi 2^3}{h} = \lim_{h \to 0} \frac{\frac{4}{3}\pi\left((2+h)^3 - 8\right)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{4}{3}\pi\left(\cancel{2^3} + 3\cdot 2^2 \cdot h + 3\cdot 2\cdot h^2 + h^3 \cancel{- 8}\right)}{h} = \lim_{h \to 0} \frac{\frac{4}{3}\pi\left(12h + 6h^2 + h^3\right)}{h}$$

$$= \frac{4}{3}\pi(12 + 0 + 0) = 16\pi.$$

§2.2 The Derivative as a Function.

Recall: $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

↓ replace a by variable x.

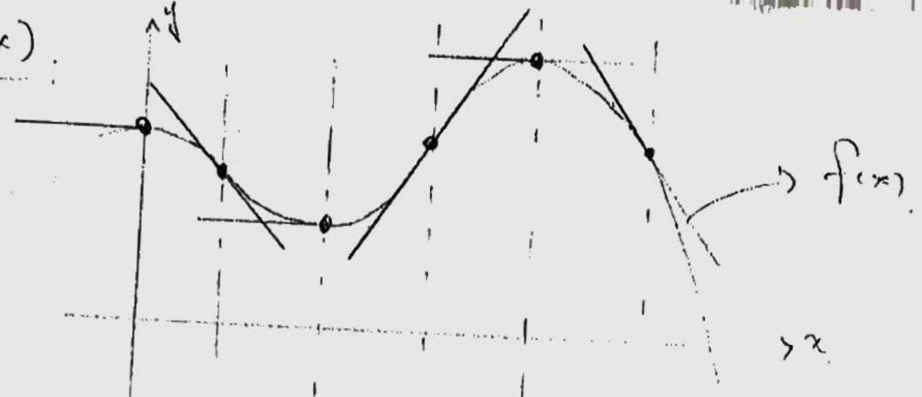
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$ derivative of $f(x)$. (or slope function).

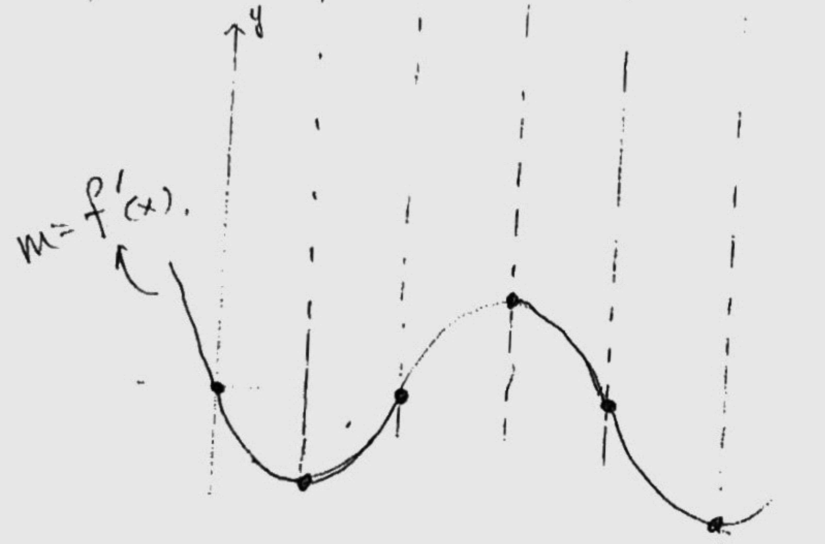
Def. A function f is called differentiable on the interval if $f'(x)$ exists at every number in that interval.

The domain of $f'(x)$ is the set $\{x \mid f'(x) \text{ exists}\}$

this domain may be different from the domain of f.

Graph of $f'(x)$.





Remark: We will see how to draw more precise graph later.

Example 1. ① Find $f'(x)$ and graph. $f(x) = 8x - 2x^2$ (parabola)

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{[8(x+h) - 2(x+h)^2] - [8x - 2x^2]}{h}$$

$$= \lim_{h \to 0} \frac{[8x + 8h - 2(x^2 + 2xh + h^2)] - [8x - 2x^2]}{h} = \lim_{h \to 0} \frac{8h - 4xh - 2h^2}{h} = 8 - 4x - 2 \cdot 0$$

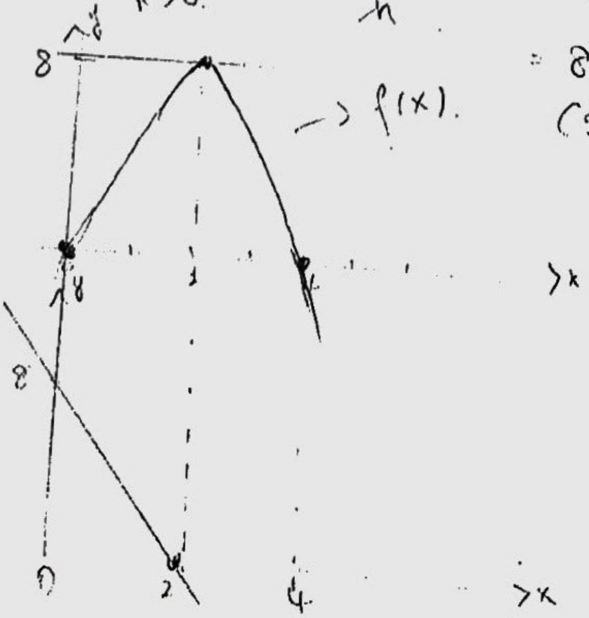
$$= 8 - 4x$$

(Straight line)

Graph

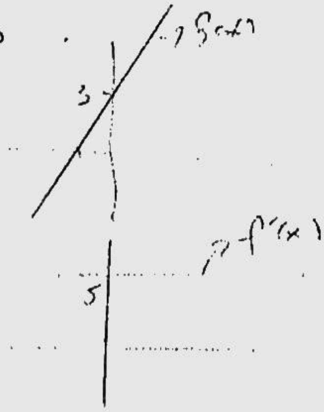
x	$f(x)$
0	0
2	$16 - 8 = 8$
4	$32 - 32 = 0$
6	$48 - 72 = -24$

x	$f'(x)$
0	$8 - 0 = 8$
2	$8 - 8 = 0$
4	$8 - 16 = -8$



③ $f(x) = 5x+3$. Since we know that for equation of line is $y = mx+b$.

$\Rightarrow m = 5$ (constant) $= f'(x)$.



Example 2. Find $f'(x)$

① $f(x) = \sqrt{x}$. $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}$

$$= \lim_{h\to 0} \frac{\sqrt{x+h}-\sqrt{x}}{h}\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) = \lim_{h\to 0} \frac{(\sqrt{x+h})^2-(\sqrt{x})^2}{h(\sqrt{x+h}+\sqrt{x})} = \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h(\sqrt{x+h}+\sqrt{x})}$$

$$= \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \frac{1}{\sqrt{x+0}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

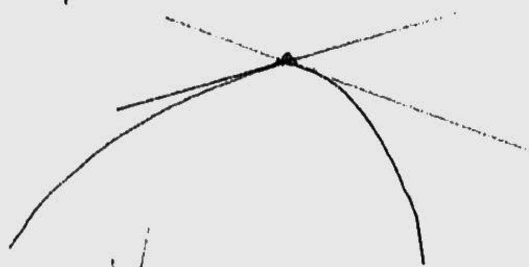
② $f(x) = \dfrac{1-x}{2+x}$. $f' = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{1-(x+h)}{2+(x+h)} - \dfrac{1-x}{2+x}}{h}$

$$= \lim_{h\to 0} \frac{\dfrac{(1-x-h)(2+x)-(1-x)(2+x+h)}{(2+x+h)(2+x)}}{h} = \lim_{h\to 0} \frac{2-2x-2h+x-x^2-xh-(2+x+h-2x-x^2-xh)}{h(2+x+h)(2+x)}$$

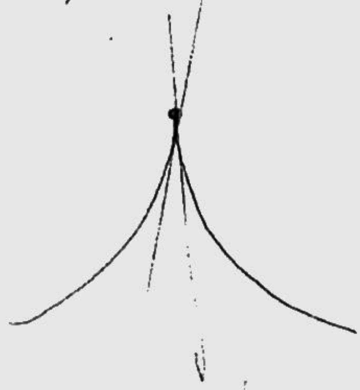
$$= \lim_{h\to 0} \frac{-2h-h}{h(2+x+h)(2+x)} = \lim_{h\to 0} \frac{-3}{(2+x+h)(2+x)} = \frac{-3}{(2+x)^2}$$

Def: Nondifferentiable functions: A function fails to have a derivative at a point. There are four cases:

① A corner.

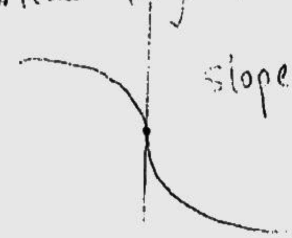


③ A cusp.

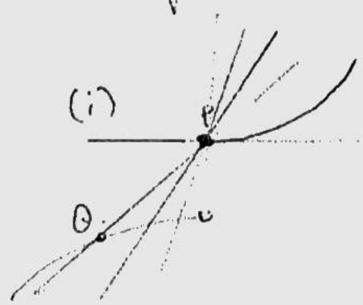


② A vertical tangent

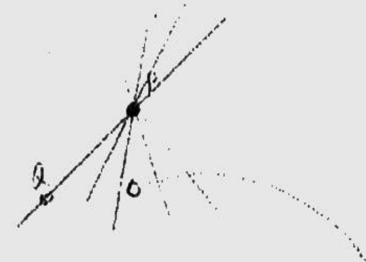
slope $\to \infty$.



④ A discontinuity. (i)

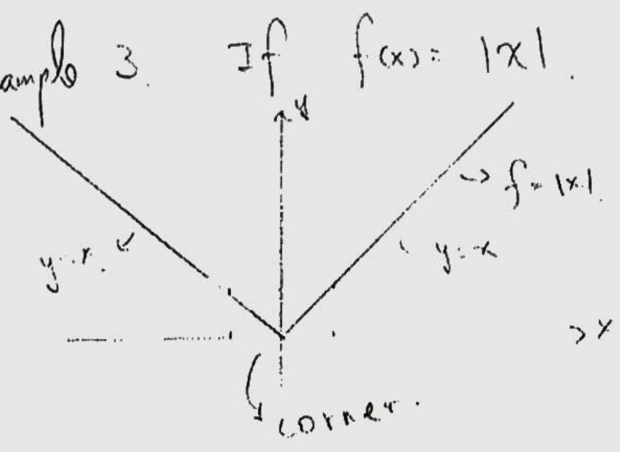


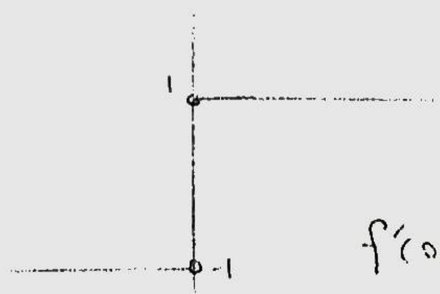
(ii)



Example 3. If $f(x) = |x|$, graph f & f'

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$





$f'(0)$ does not exist!

Theorem: If a function f is differentiable at $x = c$, then f is continuous at $x = c$.

PROOF: Given $f'(c)$ exists (from the information: f is differentiable at $x=c$)

We want to show that $\lim_{x \to c} f(x) = f(c)$

or let $x - c = h$, $x = c + h$. $\lim_{h \to 0} f(c+h) = f(c)$.

By definition $f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$.

add & subtract.

STEP 1: If $h \neq 0$.
$$
\begin{aligned}
f(c+h) &= f(c+h) + f(c) - f(c) \\
&= f(c) + \left[f(c+h) - f(c) \right] \\
&= f(c) + \left[f(c+h) - f(c) \right]\left(\frac{h}{h}\right) \\
&= f(c) + h \cdot \frac{f(c+h) - f(c)}{h}
\end{aligned}
$$

STEP 2. Now let $h \to 0$.

exists
$$
\begin{aligned}
\lim_{h \to 0} f(c+h) &= \lim_{h \to 0} \left(f(c) + h \cdot \frac{f(c+h) - f(c)}{h} \right) \\
&= f(c) + 0 \cdot f'(c) = f(c).
\end{aligned}
$$

Therefore, f is continuous at $x = c$. ☒.

Other Notations

Derivative $f'(x) = y' = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}$

$y = f(x)$ Leibniz notation.

Derivative at $x = a$ $\left.\frac{dy}{dx}\right|_{x=a} = f'(a)$.

Since $y = f(x)$. $\frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f(x)) = f'(x)$.

for example, from Example 1. $\frac{d}{dx}(8x - 2x^2) = 8 - 4x$.

Second derivatives

We call $f'(x)$ the (first) derivative of $f(x)$.

The second derivative is the derivative of $f'(x)$

that is $(f'(x))' = f''(x)$. or $\frac{d}{dx}\left(\frac{df}{dx}\right) = \frac{d^2f}{dx^2} = \frac{d^2y}{dx^2}$.

Higher derivatives.

For the nth derivative, we write $\frac{d^nf}{dx^n} = \frac{d^ny}{dx^n} = f^{(n)}(x)$.

Example 4. Find $f''(x)$ $f(x) = x^2 - 8x + 9$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left((x+h)^2 - 8(x+h) + 9\right) - \left(x^2 - 8x + 9\right)}{h}$$

$$= \lim_{h \to 0} \frac{\left(x^2 + 2xh + h^2 - 8x - 8h + 9\right) - \left(x^2 - 8x + 9\right)}{h} = \lim_{h \to 0} \frac{2xh + h^2 - 8h}{h} = 2x + 0 - 8$$

$= 2x - 8$.

linear function
$m = 2$

$\Rightarrow f''(x) = 2$. $\Rightarrow$ 2nd derivative of any quadratic equation is a constant.

Example 5. Position ($s(t)$), Velocity ($v(t)$), Acceleration ($a(t)$)

The rate of change of velocity is called the "acceleration"

$$a(t) = v'(t) = s''(t)$$
$$= \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Free fall equation. $s(t) = 4.9t^2$ (meter), from example before, we know

$v(t) = \underline{9.8}t \left(\frac{\text{meter}}{\text{sec}}\right)$ $a(t) = 9.8 \frac{\text{meter}}{\text{sec}^2}$

linear.

§2.3 Differentiation Formulas

1). $\frac{d}{dx}(c) = 0$ $y = \underline{0}x + c$
(c: const) m.

2) Observe: from Example 1 of §2.2. $f(x) = 8x - 2x^2$

$$\frac{d}{dx}(8x - 2x^2) = 8 - 4x. = 8 \cdot 1 \cdot x^0 - 2 \cdot 2 \cdot x^1$$

$$\underset{f(x)}{\frac{d}{dx}(x^3)} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^3 - x^3}{h} = \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x^3}}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2$$

The Power Rule. $\frac{d}{dx}(x^n) = nx^{n-1}$ for any integer $n > 0$. (it is also true for all real number)