§2.4. Derivatives of Trigonometric Functions

① $f(x) = \sin \underline{x}$ → radian

identity $\sin(a+b) = \sin a \cos b + \sin b \cos a$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

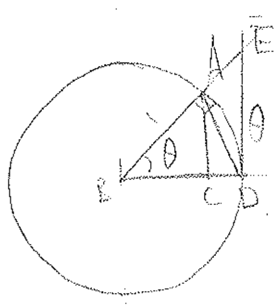
$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\left[\cos(h) - 1\right] + \cos(x)\sin(h)}{h}$$

$$= \underbrace{\lim_{h\to 0} \sin(x)}_{\sin(x)} \underbrace{\lim_{h\to 0} \frac{\cos(h)-1}{h}}_{?} + \underbrace{\lim_{h\to 0} \cos(x)}_{\cos(x)} \underbrace{\lim_{h\to 0} \frac{\sin(h)}{h}}_{= 1}$$

1 from §1.5 Example 3.

CHECK ① $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$ (IDEA: USING THE SQUEEZE Theorem)



suppose we have a unit circle (radius $r = 1$)

therefore $\overline{AB} = \overline{BD} = 1$

the arc $AD = r\theta = 1 \cdot \theta = \theta$.

Since $\sin\theta = \frac{\overline{AC}}{\overline{AB}} = \frac{\overline{AC}}{1} \Rightarrow \overline{AC} = \sin\theta$.

$\tan\theta = \frac{\overline{DE}}{\overline{BD}} = \frac{\overline{DE}}{1} \Rightarrow \overline{DE} = \tan\theta$

CHECK THE AREAS: From the graph, we can see that

↓ cont

Area of $\triangle = \frac{1}{2}\overline{BD}\cdot\overline{AC} = \frac{1}{2}\cdot 1\cdot \sin\theta = \frac{\sin\theta}{2}$

Area of sector $= \pi r^2\cdot\frac{\theta}{2\pi} = \pi\cdot 1^2\cdot\frac{\theta}{2\pi} = \frac{\theta}{2}$

Area of $\triangle = \frac{1}{2}\overline{BD}\cdot\overline{DE} = \frac{1}{2}\cdot 1\cdot\tan\theta = \frac{\tan\theta}{2}$

$\Rightarrow \quad \frac{\sin\theta}{2} < \frac{\theta}{2} < \frac{\tan\theta}{2}$ (Corus!)

$\Rightarrow \quad \sin\theta < \theta < \tan\theta$

$\Rightarrow \quad \sin\theta < \theta < \frac{\sin\theta}{\cos\theta}$

$\div \sin\theta \quad \Rightarrow \quad \frac{\sin\theta}{\sin\theta} < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

$\Rightarrow \quad 1 < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$

$\lim_{\theta\to 0} 1 = 1. \quad \lim_{\theta\to 0}\frac{1}{\cos\theta} = \frac{1}{1} = 1.$

$\Rightarrow \lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1 \Rightarrow \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1. \Rightarrow \lim_{h\to 0}\frac{\sin(h)}{h} = 1.$

② $\lim_{h\to 0}\frac{\cos(h)-1}{h} = \lim_{h\to 0}\frac{\cos(h)-1}{h}\left(\frac{\cos(h)+1}{\cos(h)+1}\right) = \lim_{h\to 0}\frac{\cos^2(h)-1}{h(\cos(h)+1)}$

since $\cos^2\theta + \sin^2\theta = 1 \Rightarrow \cos^2\theta - 1 = -\sin^2\theta.$

$= \lim_{h\to 0}\frac{-\sin^2(h)}{h(\cos(h)+1)} = \lim_{h\to 0}\frac{\sin(h)}{h}\lim_{h\to 0}\frac{-\sin(h)}{\cos(h)+1} = 1\cdot 0 = 0.$

$\Rightarrow \quad f'(x) = \sin x\cdot 0 + \cos(x)\cdot 1 = 0 + \cos x = \cos x.$

$$\boxed{\frac{d}{dx}(\sin x) = \cos x}$$

② $f(x) = \cos x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\left[\cos(h) - 1\right] - \sin(x)\sin(h)}{h}$$

$$= \underbrace{\lim_{h \to 0} \cos(x)}_{\cos(x)} \underbrace{\lim_{h \to 0} \frac{\cos(h) - 1}{h}}_{0} - \lim_{h \to 0} \sin(x) \underbrace{\lim_{h \to 0} \frac{\sin(h)}{h}}_{1}$$

$$= \cos(x) \cdot 0 - \sin x \cdot 1 = -\sin x.$$

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

Example 1. ① $y = x^2 \sin x$, find y'

$$y' = \frac{d}{dx}(x^2) \cdot \sin x + x^2 \cdot \frac{d}{dx}(\sin x)$$

$$= 2x \cdot \sin x + x^2 \cdot \cos x.$$

② $y = \sqrt{x} \cos x$. find y'

$$y' = \frac{d}{dx}(\sqrt{x}) \cos x + \sqrt{x} \frac{d}{dx}(\cos x) = \frac{1}{2\sqrt{x}} \cos x - \sqrt{x} \sin x.$$

③ $f(x) = \tan x = \dfrac{\sin x}{\cos x}$

$$f'(x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \cdot \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{\cos x \cdot \cos x + \sin x \sin x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \underline{\sec^2 x}$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

④ $$\boxed{\frac{d}{dx}(\cot x) = -\csc^2 x}$$

⑤ $f(x) = \sec x = \dfrac{1}{\cos x}$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(\cos x)}{(\cos x)^2}$$

$$= \frac{0 + \sin x}{\cos^2 x} = \frac{\sin x}{\cos^2 x} = \frac{1}{\cos x} \cdot \frac{\sin x}{\cos x}$$

$$= \sec x \cdot \tan x$$

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

⑥ $$\boxed{\frac{d}{dx}(\csc x) = -\csc x \cot x}$$

P.4

Example 2. $f(x) = \dfrac{\sec x}{1+\tan x}$ find $f'(x)$.

$$f'(x) = \frac{d}{dx}\left(\frac{\sec x}{1+\tan x}\right) = \frac{(1+\tan x)\frac{d}{dx}(\sec x) - \sec x \frac{d}{dx}(1+\tan x)}{(1+\tan x)^2}$$

$$= \frac{(1+\tan x)(\sec x \tan x) - \sec x(0+\sec^2 x)}{(1+\tan x)^2}$$

$\tan^2 x + 1 = \sec^2 x$.

$$= \frac{\sec x \tan x + \sec x \tan^2 x - \sec^3 x}{(1+\tan x)^2} = \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$$

$$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}.$$

Example 3. Find $f^{(27)}(x)$ if $f(x) = \cos x$.

derivative cycle

1^{st} $f'(x) = -\sin x$. 5^{th} $f^{(5)} = -\sin x$

2^{nd} $f''(x) = -\cos x$. 6^{th}

3^{rd} $f'''(x) = +\sin x$. 7^{th} 27^{th}

4^{th} $f^{(4)}(x) = \cos x$ 8^{th} 12^{th}, 16^{th} 20^{th} 24^{th} 28^{th}

$\Rightarrow$ $f^{(27)}(x) = \sin x$

Example 4. ① Find $\lim_{x\to 0} \frac{\sin(7x)}{4x}$ IDEA $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$.

$\theta = 7x$.

$$\lim_{x\to 0} \frac{\sin(7x)}{4x} = \lim_{x\to 0} \frac{\sin(7x)}{7x} \cdot \frac{7x}{4x} = \lim_{x\to 0} \frac{\sin(7x)}{7x} \cdot \lim_{x\to 0} \frac{7x}{4x} = 1 \cdot \frac{7}{4} = \frac{7}{4}.$$

②
$$\lim_{x\to 0} x\cot x = \lim_{x\to 0} x \cdot \frac{\cos x}{\sin x} = \underbrace{\lim_{x\to 0} \frac{x}{\sin x}}_{1} \cdot \lim_{x\to 0} \cos x = 1 \cdot \cos 0 = 1.$$

§2.5 Chain Rule

IDEA: Can we differentiate a composite function $(f\circ g)(x) = f(g(x))$.

Consider $F(x) = \sqrt{x^2+1}$ find $F'(x)$

let $f(x) = \sqrt{x}$, $g(x) = x^2+1$ $\xrightarrow[\text{CHECK}]{}$ $f(g(x)) = f(x^2+1) = \sqrt{x^2+1} = F(x)$ ✓.

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f(g(x))$ is differentiable at x.

and $F'(x)$ is given by

$$F'(x) = f'(g(x)) \cdot g'(x).$$

If we let $u = g(x)$, $y = F = f(u)$, then.

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$