§3.3. <u>The Second Derivative Test</u>

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $c$.


Concave up

(b) If $f'(c) = 0$, and $f''(c) < 0$ then $f$ has a local maximum at $c$.

Concave down

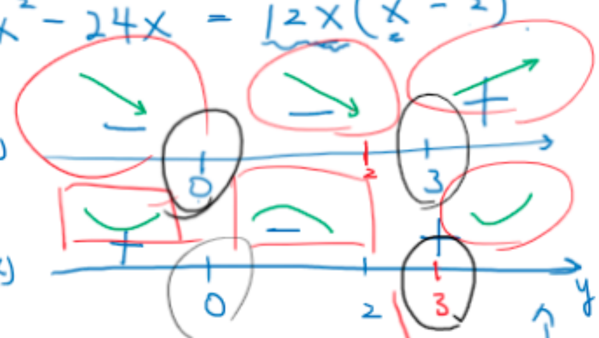
Example sketch $y = x^4 - 4x^3 = f(x)$

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

$$f''(x) = 12x^2 - 24x = 12x(x-2)$$

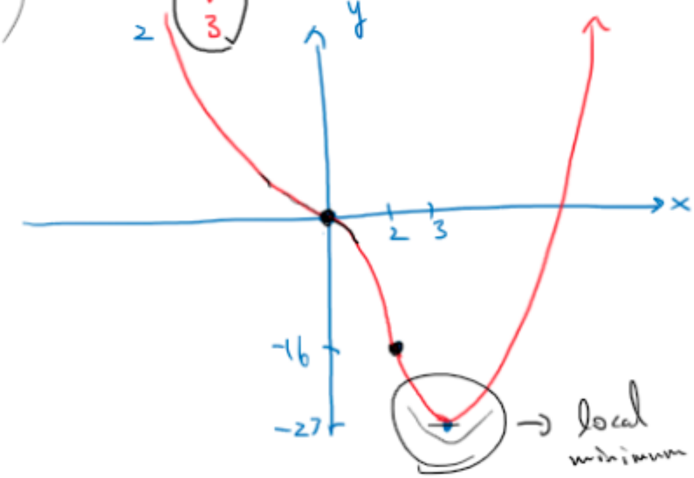
$f'(x) = 0$ if $x = 0, 3$

$f''(x) = 0$ if $x = 0, 2$



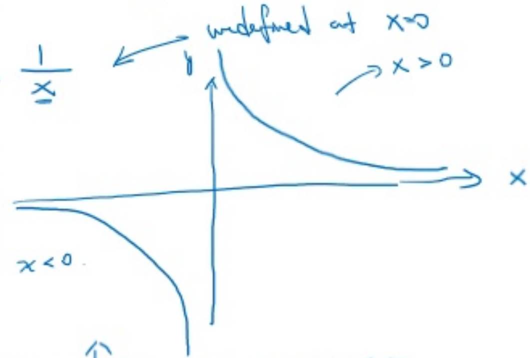

local minimum at $x = 3$

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 2 | $2^4 - 4\cdot 2^3 = 16 - 32 = -16$ |
| 3 | $3^4 - 4\cdot 3^3 = 81 - 4\cdot 27 = -27$ |



# §3.4 Limits at Infinity: Horizontal Asymptotes.

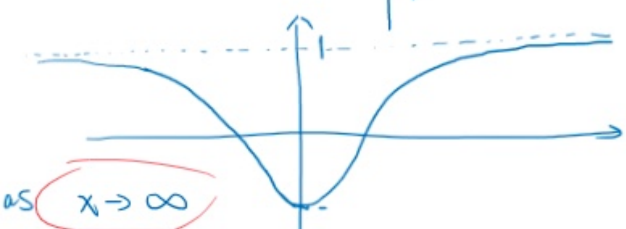
Consider the cases ① $f(x) = \frac{1}{x}$




$f(x)$ goes to $0$ as $x \to \infty$

$f(x)$ goes to $0$ as $x \to -\infty$

② $f(x) = \frac{x^2-1}{x^2+1}$




$f(x)$ goes to $1$ as $x \to \infty$

$f(x)$ goes to $1$ as $x \to -\infty$

Def. We say that $f(x)$ has the limit $L$ as $x$ approaches "infinity"

and we write $$\lim_{x \to \infty} f(x) = L$$

Similarly, for the case $f(x) \to L$ as $x \to -\infty$,

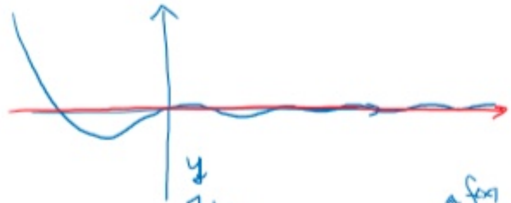
we write $$\lim_{x \to -\infty} f(x) = L.$$

When we have the vertical asyptotes we have $\lim_{x \to a} f(x) = \pm\infty$.
$\downarrow$
$x = a$ is the vertical asymptote.

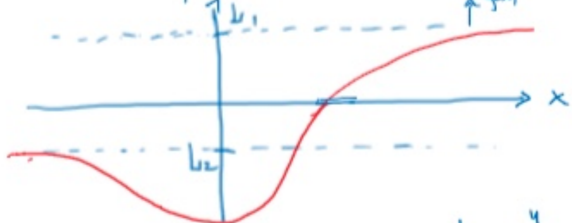
Def. The line $y = L$ is called a horizontal asymptote of $y = f(x)$ if

$$\lim_{x \to \infty} f(x) = L \text{ "or" } \lim_{x \to -\infty} f(x) = L$$

for example. ①



②




$\lim_{x\to\infty} f(x) = L_1$

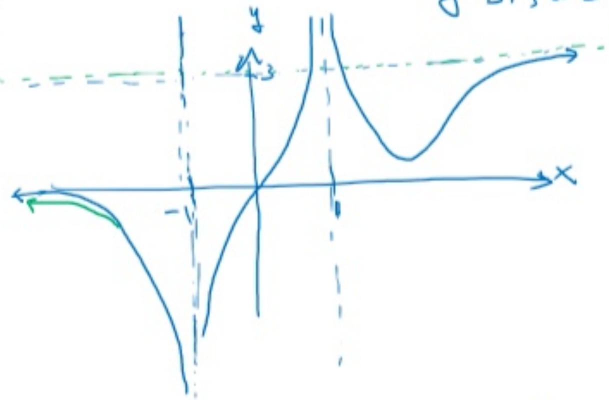
$\lim_{x\to-\infty} f(x) = L_2$ $\Rightarrow$ 2 horizontal asymptotes

$y = L_1$, and $y = L_2$

Example 1. ① Find asymptotes

Vertical asymptotes $\begin{cases} x = 1 \\ x = -1 \end{cases}$

Horizontal asymptotes $\begin{cases} y = 3 \\ y = 0 \end{cases}$ ✓




② Evaluate limits.

$$\lim_{x\to\infty} \frac{1}{x} = 0, \quad \lim_{x\to-\infty} \frac{1}{x} = 0, \quad \lim_{x\to\infty} \frac{1}{x^r} = 0, \quad r > 0$$

Example 2 Evaluate $\lim_{x\to\infty} \dfrac{3x^2 - x - 2}{5x^2 + 4x + 1}$ ?

$\dfrac{\div x^2}{\div x^2}$

$$\lim_{x\to\infty} \frac{3\cancel{\frac{x^2}{x^2}} - \cancel{\frac{x^1}{x^2}} - \frac{2}{x^2}}{5\cancel{\frac{x^2}{x^2}} + 4\frac{x^1}{x^2} + \frac{1}{x^2}} = \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + 4\frac{1}{x} + \frac{1}{x^2}} = \boxed{\frac{3}{5}}$$