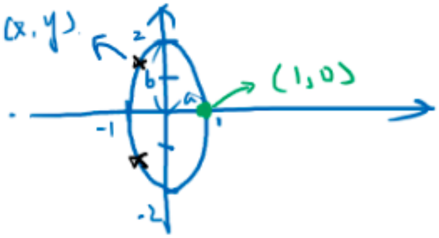
$\to y^2 = 4 - 4x^2$

$\left(-\frac{1}{3}, \pm\frac{4\sqrt{2}}{3}\right)$

23. Find the points on the ellipse $4x^2 + y^2 = 4$ that are farthest away from the point (1, 0).

$\div 4$

$x^2 + \frac{y^2}{4} = 1 \Rightarrow \frac{x^2}{\underset{a}{1^2}} + \frac{y^2}{\underset{b}{2^2}} = 1$



distance $d = \sqrt{(x-1)^2 + (y-0)^2}$

$d^2 = (x-1)^2 + y^2 = (x-1)^2 + 4 - 4x^2$

$d^2 = f(x) = (x-1)^2 + 4 - 4x^2$

$f'(x) = 2(x-1) - 8x$, $\quad f'(x) = 0 \Rightarrow$

$2(x-1) - 8x = 0$

$-6x - 2 = 0 \quad -6x = 2$

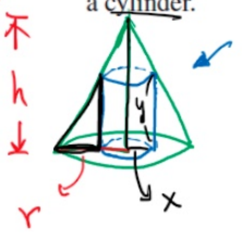
$\boxed{x = \frac{-1}{3}}$

$y^2 = 4 - 4x^2 = 4 - 4\left(\frac{-1}{3}\right)^2 = 4 - 4\left(\frac{1}{9}\right) = \frac{32}{9}$ $\quad 36-4$

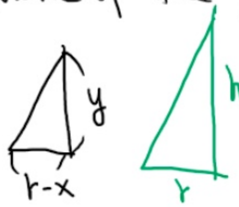
$\boxed{y = \pm\sqrt{\frac{32}{9}} = \pm\frac{4\sqrt{2}}{3}}$

$\left(\frac{4\pi r^2 h}{27} \right)$

32. A right circular cylinder is inscribed in a cone with height *h* and base radius *r*. Find the largest possible volume of such a cylinder.



volume of the cylinder $= \pi x^2 y$



$\Rightarrow$ from the similar triangle

$$\frac{h}{r} = \frac{y}{r-x}$$

$$\Rightarrow y = \frac{h(r-x)}{r}$$

$$V(x) = \pi x^2 \frac{h(r-x)}{r} = \frac{\pi h}{r}(rx^2 - x^3)$$

$$V'(x) = \frac{\pi h}{r}(2rx - 3x^2),$$

$\Rightarrow V'(x) = 0 \Rightarrow 2rx - 3x^2 = 0$

$\Rightarrow x(2r - 3x) = 0$

$\Rightarrow 2r = 3x, \; x = \frac{2r}{3}.$ $\quad y = \frac{h(r - \frac{2r}{3})}{r} = \frac{h\frac{1}{3}r}{r} = \frac{h}{3}$

$$V = \pi \left(\frac{2r}{3}\right)^2 \frac{h}{3} = \frac{4\pi r^2 h}{27} \; ☆$$

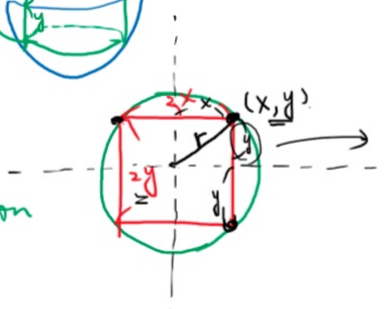
$\pi r^2(1+\sqrt{5})$

33. A right circular cylinder is inscribed in a sphere of radius *r*. Find the largest possible surface area of such a cylinder.

surface area of a cylinder $= 2\pi x^2 + 2\pi x 2y$

$= 2\pi x^2 + 4\pi x y$

Cross-section →



$x^2+y^2=r^2 \Rightarrow y = \pm\sqrt{r^2-x^2} = \sqrt{r^2-x^2}$

pick "+"

$$S(x) = 2\pi x^2 + 4\pi x\sqrt{r^2-x^2}$$

$$S'(x) = 4\pi x + 4\pi\sqrt{r^2-x^2} + 4\pi x \frac{-\cancel{2}x}{\cancel{2}\sqrt{r^2-x^2}} = 4\pi x + 4\pi\sqrt{r^2-x^2} - \frac{4\pi x^2}{\sqrt{r^2-x^2}}$$

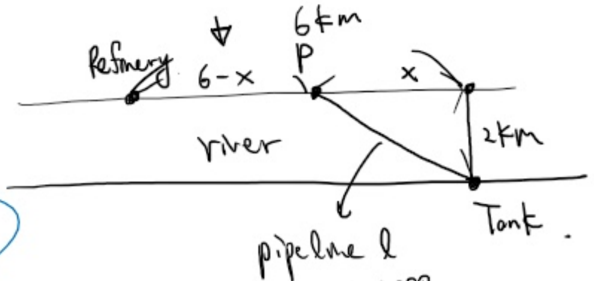
$$\Rightarrow S'(x) = 0 \Rightarrow \cancel{4\pi}x + \cancel{4\pi}\sqrt{r^2-x^2} - \frac{\cancel{4\pi}x^2}{\sqrt{r^2-x^2}} = 0 \quad \cdots \boxed{x = \sqrt{\frac{5+\sqrt{5}}{10}}\, r}$$

≈4.85 km east of the refinery.

51. An oil refinery is located on the north bank of a straight river that is 2 km wide. A pipeline is to be constructed from the refinery to storage tanks located on the south bank of the river 6 km east of the refinery. The cost of laying pipe is \$400,000/km over land to a point *P* on the north bank and \$800,000/km under the river to the tanks. To minimize the cost of the pipeline, where should *P* be located?

$$6 - x = \boxed{6 - \frac{2}{\sqrt{3}}}$$

$$\approx \underline{4.8453} \text{ km} \checkmark$$



$$l^2 = x^2 + 2^2$$

$$\text{Cost } C = (6-x) \cdot \overset{\to 400{,}000}{4\cancel{00000}} + l \times \overset{\to 80000}{8\cancel{00000}}$$

$$= (6-x)4 + \sqrt{x^2+2^2} \cdot 8$$

$$C(x) = 24 - 4x + 8\sqrt{x^2+4}$$

$$C'(x) = -4 + 8\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+4}} = -4 + \frac{8x}{\sqrt{x^2+4}}$$

$$\text{set } C'(x) = 0 \Rightarrow -4 + \frac{8x}{\sqrt{x^2+4}} = 0$$

$$\Rightarrow \frac{-4\sqrt{x^2+4} + 8x}{\sqrt{x^2+4}} = 0 \Rightarrow \overset{2}{\cancel{8}}x = \cancel{4}\sqrt{x^2+4}$$

square both sides $\Rightarrow 4x^2 = x^2 + 4$

$$\Rightarrow 3x^2 = 4, \quad x^2 = \frac{4}{3}$$

$$x = \frac{2}{\sqrt{3}}$$