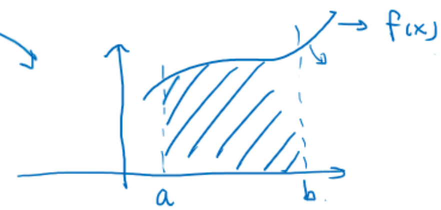
11/06.

Chapter 4. Integrals

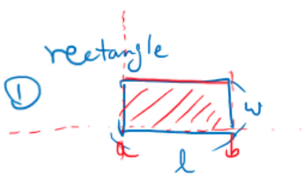
Goal: Find the area of the region bounded by the curve $f(x)$ and the x-axis.



§4.1. Areas and Distances

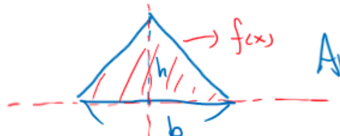
We have seen:

① rectangle



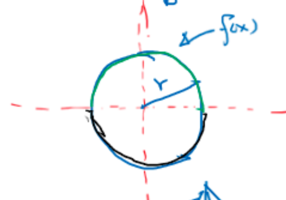
Area: wl

② Triangle



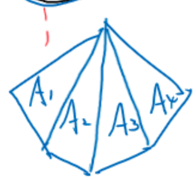
Area: $\frac{1}{2}bh$

③ Circle



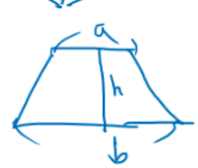
Area: πr^2

④ polygon



Area $= A_1 + A_2 + A_3 + A_4$

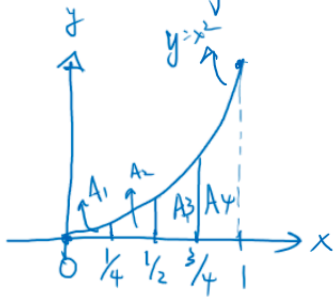
⑤ Trapezoid



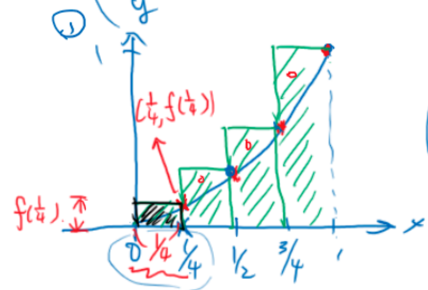
Area $= \frac{1}{2}(a+b)h$

Example 1. Estimate the area under the parabola $y = x^2 = f(x)$

from 0 to 1. (exact area is $A = A_1 + A_2 + A_3 + A_4$)



①



right-endpoint estimation.

(using right-endpoints will <u>overestimate</u> the exact area)

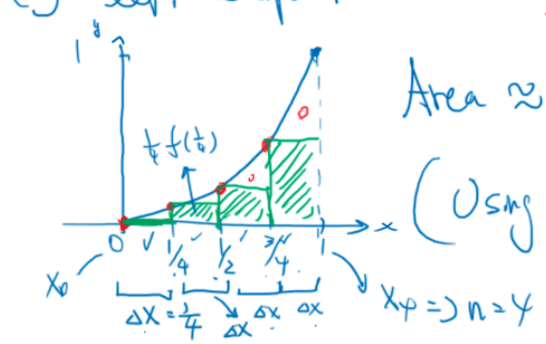
$$\text{Area} \approx \frac{1}{4} \cdot f\left(\frac{1}{4}\right) + \frac{1}{4} f\left(\frac{1}{2}\right) + \frac{1}{4} f\left(\frac{3}{4}\right) + \frac{1}{4} f(1)$$

② left-endpoint

$\Delta x \cdot f(0) + \Delta x \cdot f(\ \ ) + \cdots$

$$\text{Area} \approx \underline{\underline{\frac{1}{4}}} \cdot f(0) + \underline{\underline{\frac{1}{4}}} f\left(\frac{1}{4}\right) + \underline{\underline{\frac{1}{4}}} f\left(\frac{1}{2}\right) + \underline{\underline{\frac{1}{4}}} f\left(\frac{3}{4}\right)$$

(Using left-endpoints will <u>underestimate</u> the area)

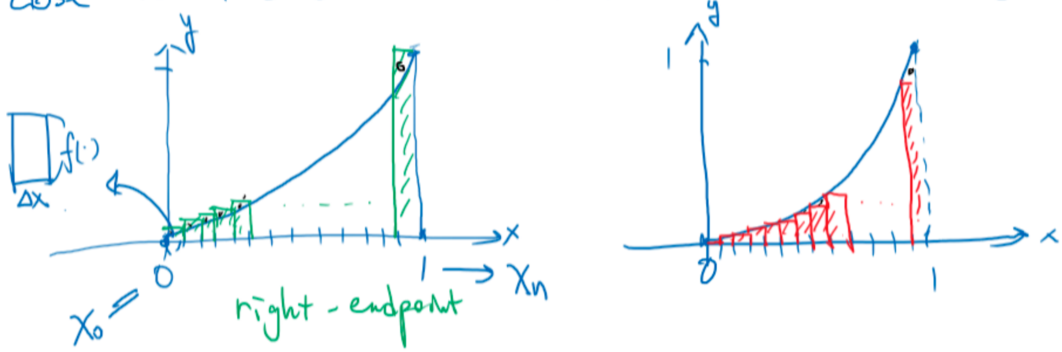


What if we have a "finer" discretization.

discretization : we divide an interval into many subintervals.
and each subinterval shares the same length Δx.

also, $\Delta x = \dfrac{b-a}{n}$

We can expect that the right/left-endpoint areas will be close to the exact area A when Δx is (very) small.



If we let $x_0 = 0$, $x_n = 1$, then we have n rectangles.

① right-endpoint

$$R_n = \Delta x \cdot \underbrace{f(x_1)} + \Delta x \cdot f(x_2) + \cdots \Delta x \cdot f(x_n)$$

$$f(x) = x^2 \qquad = \Delta x \left(f(x_1) + f(x_2) + \cdots + f(x_n) \right)$$

$$x_1 = x_0 + \Delta x \qquad = \frac{1}{n}\left(f\left(\tfrac{1}{n}\right) + f\left(\tfrac{2}{n}\right) + \cdots + f\left(\tfrac{n}{n}\right) \right)$$

$$x_2 = x_0 + 2\Delta x$$

$$\vdots$$

$$= \frac{1}{n}\left(\left(\tfrac{1}{n}\right)^2 + \left(\tfrac{2}{n}\right)^2 + \cdots + \left(\tfrac{n}{n}\right)^2 \right)$$

$$x_i = x_0 + i\Delta x = 0 = \frac{i}{n}$$

$i = 0, \cdots, n$

$$= \underbrace{\frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2 \right)}$$

② left-endpoint

$$L_n = \Delta x\, f(x_0) + \Delta x\, f(x_1) + \cdots + \Delta x \cdot f(x_{n-1})$$

$$= \frac{1}{n}\left(f(0) + f\left(\frac{1}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right)$$

$$= \frac{1}{n}\left(0^2 + \left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{n-1}{n}\right)^2\right)$$

$$= \frac{1}{n^3}\left(0^2 + 1^2 + \cdots (n-1)^2\right)$$

$$\Rightarrow \quad R_n = \frac{1}{n^3} \cdot \frac{n(n+1)(2n+1)}{6}$$

$$L_n = \frac{1}{n^3} \cdot \frac{(n-1)(n)(2n-1)}{6}$$

Take limit as $n \to \infty$ $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{3}$.

$2n^3 + \cdots \quad \frac{n^2}{n^3} = \frac{1}{n}$

$\rightarrow 2n^3 + \cdots$

$\lim_{n\to\infty} L_n = \lim_{n\to\infty} \frac{(n-1)\,n\,(2n-1)}{6n^3} = \frac{2}{6} = \frac{1}{3}$.

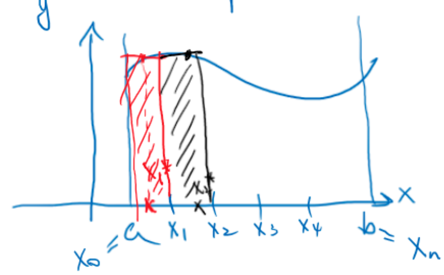
$$L_n \le A \le R_n$$

$$\Rightarrow \underbrace{\lim_{n\to\infty} L_n}_{= \frac{1}{3}} \le A \le \underbrace{\lim_{n\to\infty} R_n}_{= \frac{1}{3}} \Rightarrow \text{Area } A = \frac{1}{3}.$$

Def. The area A of the region that lies under the graph of $f(x)$ is the limit of the sums of the areas of approximating rectangles.

$$A = \lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(f(x_1)\Delta x + \cdots + f(x_n)\Delta x \right)$$

Remark: We can pick points other than endpoints and we call them sample points x_i^*



$$A = \lim_{n\to\infty}\left(f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x \right)$$