CONDITIONAL PROBABILITY:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

GENERAL MULTIPLICATION RULE:

$$P(A \cap B) = P(A)P(B|A)$$
$$= P(B)P(A|B)$$

USE INFO ON THE RIGHT OF EACH EQUATION TO FIND PROBABILITY ON THE LEFT OF THE EQUATION

4.46 Dice An experiment consists of tossing a single die and observing the number of dots that show on the upper face. Events *A*, *B*, and *C* are defined as follows:

A: Observe a number less than 4

B: Observe a number less than or equal to 2

C: Observe a number greater than 3

Find the probabilities associated with the events below using either the simple event approach or the rules and definitions from this section.

a. S	**b.** $A \mid B$	**c.** B
d. $A \cap B \cap C$	**e.** $A \cap B$	**f.** $A \cap C$
g. $B \cap C$	**h.** $A \cup C$	**i.** $B \cup C$

(a) $P(S) = 1$ BY DEFINITION

(b) $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{P(1 \text{ or } 2)}{P(1 \text{ or } 2)} = 1$

PROB # ≤ 4 GIVEN # ≤ 2

§4.7 BAYES' RULE

1. LAW OF TOTAL PROBABILITY
2. BAYES' RULE

3. COMBINED

1. Law of Total Probability.

Suppose sample space Ω is broken up (partitioned) into k subspaces,

$$S_1, S_2, S_3, \ldots, S_k$$

such that the subspaces are

(1) **Mutually exclusive**

$$P(S_i \cap S_j) = 0, \quad i \neq j$$

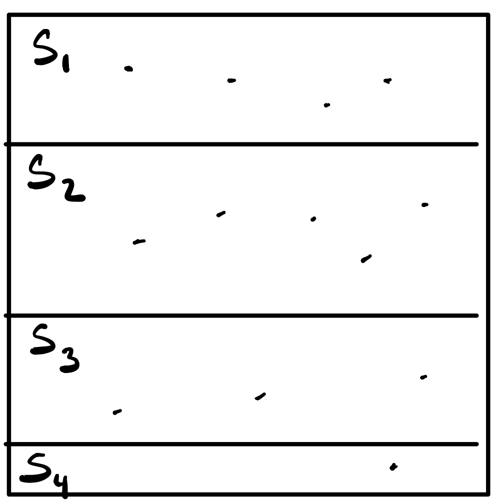
(no overlap between subspaces)

(2) **Exhaustive**

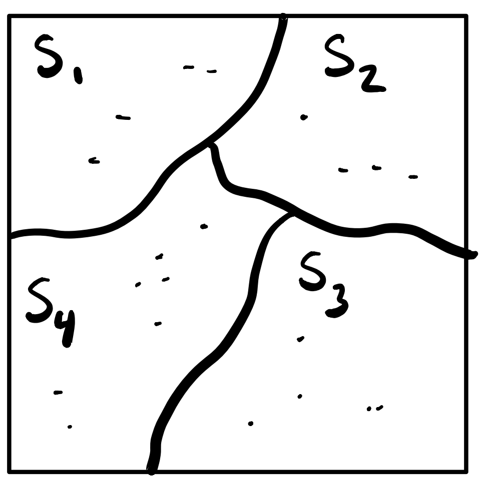
Equivalent
$$\begin{cases} S_1 \cup S_2 \cup S_3 \cup \ldots \cup S_k = \Omega \\ P(S_1) + P(S_2) + P(S_3) + \ldots + P(S_k) = 1 \end{cases}$$

e.g.

Ω



or



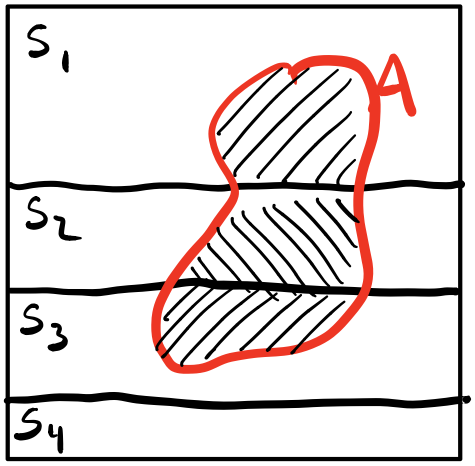
Ω

* **Every simple event is inside 1 and only 1 subspace.**

$\underbrace{\text{inside 1}}_{\text{exhaustive}}$ $\underbrace{\text{and only 1}}_{\text{mutually exclusive}}$

LAW OF TOTAL PROBABILITY:

IF $S_1, \dots, S_K$ ARE MUTUALLY EXCLUSIVE & EXHAUSTIVE, THEN FOR ANY EVENT A,



$$A = (A \cap S_1) \cup (A \cap S_2) \cup \dots \cup (A \cap S_K)$$

NOTE SOME INTERSECTIONS MAY BE EMPTY. IT'S OK.
$(A \cap S_4 = \phi)$

$$\Rightarrow P(A) = P\big((A \cap S_1) \cup (A \cap S_2) \cup \dots \cup (A \cap S_K)\big)$$

MUTUALLY EXCLUSIVE

$$P(A) = P(A \cap S_1) + P(A \cap S_2) + \dots + P(A \cap S_K)$$

GENERAL RULE OF MULT. $P(A \cap S_i) = P(S_i)P(A|S_i)$

$$P(A) = P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + \dots + P(S_K)P(A|S_K)$$

LAW OF TOTAL PROBABILITY

ex. Population of NYC residents is naturally broken into subpopulations:

19 % of NYC residents live in Manhattan
31 % of NYC residents live in Brooklyn
27 % of NYC residents live in Queens
17 % of NYC residents live in Bronx
6 % of NYC residents live in Staten Island

(Subspace / subpopulation interchangeably)

29% of Manhattan residents are foreign born
38% of Brooklyn residents are foreign born
49 % of Queens residents are foreign born
32 % of Bronx residents are foreign born
21 % of S.I. residents are foreign born

F = Foreign born

$P(F \mid M) = .29$
$P(F \mid B) = .38$
$P(F \mid Q) = .49$
$P(F \mid BX) = .32$
$P(F \mid SI) = .21$

Use this to find unconditional prob. $P(F)$.

LAW OF TOTAL PROB: $P(F)$ IS A WEIGHTED AVERAGE OF $P(F|M), P(F|BK), \ldots, P(F|SI)$.

$$\left.\begin{aligned} P(F) = {} & .2\,P(F|M) + .2\,P(F|BK) + .2\,P(F|Q) \\ & + .2\,P(F|BX) + .2\,P(F|SI) \end{aligned}\right\} \text{WEIGHTED EQUALLY}$$

$$\left.\begin{aligned} P(F) = {} & P(M)P(F|M) + P(BK)P(F|BK) + P(Q)P(F|Q) \\ & + P(BX)P(F|BX) + P(SI)P(F|SI) \end{aligned}\right\} \text{WEIGHTED ACCORDING TO } P(\text{BOROUGH})$$

$$= .19(.29) + .31(.38) + .27(.49) + .17(.32) + .06(.21)$$

$$= .3722$$

ex. SUPPOSE 30% OF SHOPPERS AT STORE X ARE AGE ≤ 20.
SUPPOSE 50% OF SHOPPERS AT STORE X ARE AGE 21 - 40.
SUPPOSE 20% OF SHOPPERS AT STORE X ARE AGE ≥ 41.
IF 80% OF SHOPPERS AGE ≤ 20 PLAY VIDEO GAMES,
40% OF SHOPPERS 21 - 40 PLAY VIDEO GAMES,
10% OF SHOPPERS AGE ≥ 41 PLAY VIDEO GAMES.

WHAT PERCENTAGE OF SHOPPERS PLAY VIDEO GAMES.

LET S_1 = SHOPPERS AGE ≤ 20
S_2 = SHOPPERS AGE 21-40
S_3 = SHOPPERS AGE ≥ 41

A = SHOPPER PLAYS VIDEO GAMES.

GIVEN: $P(S_1) = .3$ $\quad P(A|S_1) = .8$
$P(S_2) = .5$ $\quad P(A|S_2) = .4$
$P(S_3) = .2$ $\quad P(A|S_3) = .1$

FIND $P(A) = P(S_1)P(A|S_1) + P(S_2)P(A|S_2) + P(S_3)P(A|S_3)$
$= (.3)(.8) + (.5)(.4) + (.2)(.1)$
$= .46$ WEIGHTED AVERAGE OF CONDITIONAL PROB.

ex. 80% OF PEOPLE WHO OWN CARS GO ON VACATION AT LEAST ONCE A YEAR.

65% OF PEOPLE WHO DO NOT OWN CARS GO ON VACATION AT LEAST ONCE A YEAR.

SUPPOSE 58% OF PEOPLE OWN A CAR.

WHAT PERCENT OF PEOPLE GO ON VACATION AT LEAST ONCE A YEAR?

HAS THE POPULATION BEEN BROKEN INTO SUBPOPULATIONS THAT ARE MUTUALLY EXCLUSIVE & EXHAUSTIVE?

LET S_1 = PERSON OWNS A CAR
S_2 = PERSON DOES NOT OWN A CAR.
A = PERSON GOES ON VACATION AT LEAST ONCE A YEAR.

GIVEN: $P(S_1) = .58$

$P(S_2) = P(S_1^c) = 1 - P(S_1) = .42$

$P(A|S_1) = .8$

$P(A|S_2) = .65$

FIND $P(A)$.

LOTP: $P(A) = P(S_1)P(A|S_1) + P(S_2)P(A|S_2)$

$= (.58)(.8) + (.42)(.65)$

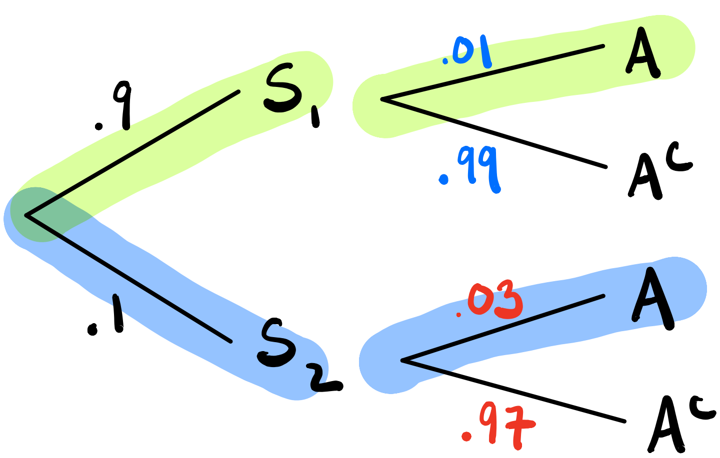
$= .737 \rightarrow \boxed{73.7\%}$

4.73 Worker Error A worker-operated machine produces a defective item with probability .01 if the worker follows the machine's operating instructions exactly, and with probability .03 if he does not. If the worker follows the instructions 90% of the time, what proportion of all items produced by the machine will be defective?

NOTATION: S_1 = WORKER FOLLOWS INSTRUCTIONS

S_2 = WORKER DOES NOT FOLLOW INSTRUCTIONS

A = MACHINE PRODUCES DEFECTIVE ITEM.



FIND $P(A)$

$P(A) = P(A \cap S_1) + P(A \cap S_2)$

$$P(A) = (.9)(.01) + (.1)(.03) = .012$$

$$\left(P(A) = P(S_1)P(A|S_1) + P(S_2)P(A|S_2) \quad \text{LoTP} \right)$$

GENERAL MULTIPLICATION RULE

BAYES' RULE:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \begin{cases} \dfrac{P(A)P(B|A)}{P(B)} & \text{BAYES' RULE!} \\[2ex] \dfrac{\cancel{P(B)}P(A|B)}{\cancel{P(B)}} & \text{NOT USEFUL} \end{cases}$$

$$\text{BAYES' RULE:} \quad P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}$$

OPPOSITE CONDITIONAL PROBABILITIES

ex.

SUPPOSE 30% OF SHOPPERS AT STORE X ARE AGE ≤ 20.

SUPPOSE 50% OF SHOPPERS AT STORE X ARE AGE 21 - 40.

SUPPOSE 20% OF SHOPPERS AT STORE X ARE AGE ≥ 41.

IF 80% OF SHOPPERS AGE ≤ 20 PLAY VIDEO GAMES,

40% OF SHOPPERS 21 - 40 PLAY VIDEO GAMES,

10% OF SHOPPERS AGE ≥ 41 PLAY VIDEO GAMES.

GIVEN THAT A SHOPPER PLAYS VIDEO GAMES, WHAT IS THE PROBABILITY THAT THE SHOPPER IS AGE ≤ 20 ?

LET S_1 = SHOPPERS AGE ≤ 20
S_2 = SHOPPERS AGE 21-40
S_3 = SHOPPERS AGE ≥ 41

A = SHOPPER PLAYS VIDEO GAMES.

FIND $P(S_1 | A)$

GIVEN: $P(S_1) = .3$ $\quad P(A|S_1) = .8$
$P(S_2) = .5$ $\quad P(A|S_2) = .4$
$P(S_3) = .2$ $\quad P(A|S_3) = .1$

OPPOSITE CONDITIONAL PROBABILITIES!

$\Rightarrow$ USE BAYES' RULE!

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)} \quad : \quad P(S_1|A) = \frac{P(S_1)P(A|S_1)}{P(A)} = \frac{(.3)(.8)}{(.46)} \leftarrow \text{LoTP}$$

TURN A INTO S_1
B INTO A

$$= \boxed{.5217}$$

<u>ex.</u> SUPPOSE A TEST FOR A RARE DISEASE IS 99% ACCURATE:

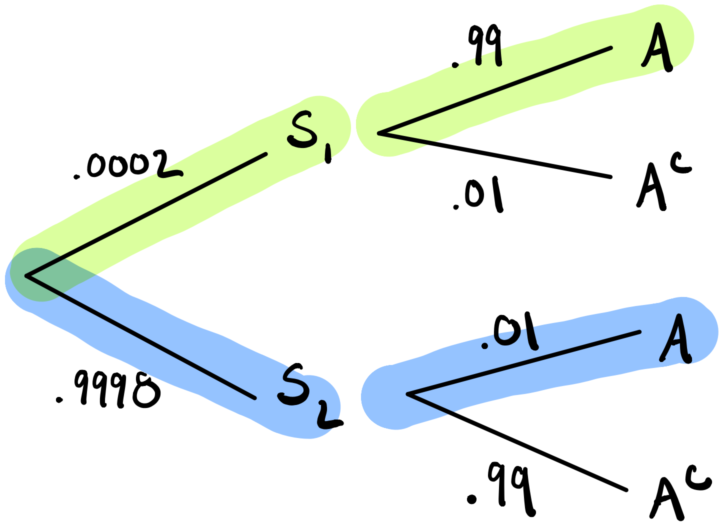
99% OF PEOPLE WITH THE DISEASE TEST POSITIVE.

99% OF PEOPLE WITHOUT THE DISEASE TEST NEGATIVE.

IF .02% OF POPULATION HAS THE DISEASE, FIND THE PROB. THAT A PERSON WHO TESTS POSITIVE HAS THE DISEASE.

LET S_1 = DISEASE
S_2 = NO DISEASE
A = TEST POSITIVE

GIVEN $P(A|S_1) = .99$
$P(A^c|S_2) = .99$
$P(S_1) = .0002$
$P(S_2) = .9998$



FIND $P(S_1 | A) \overset{BR}{=} \dfrac{P(S_1)P(A|S_1)}{P(A)} \overset{LoTP}{=} \dfrac{P(S_1)P(A|S_1)}{P(S_1)P(A|S_1) + P(S_2)P(A|S_2)}$

$$= \frac{(.0002)(.99)}{(.0002)(.99) + (.9998)(.01)} = \boxed{.0194}$$

OR $\approx 2\%$