§ 4.6 INDEPENDENCE, CONDITIONAL PROBABILITY, & THE MULTIPLICATION RULE.

<u>Def</u>: TWO EVENTS ARE SAID TO BE <u>INDEPENDENT</u> IF THE PROBABILITY OF ONE EVENT A IS NOT INFLUENCED OR CHANGED BY THE OCCURRENCE OF THE OTHER EVENT B, AND VICE VERSA.

<u>ex.</u> $\left.\begin{array}{l}\text{A: IT IS MONDAY.}\\ \text{B: IT RAINS.}\end{array}\right\}$ INDEPENDENT EVENTS.

<u>ex.</u> EXPERIMENT: FLIP TWO COINS.

$\left.\begin{array}{l}\text{A: } 1^{st} \text{ COIN IS HEADS}\\ \text{B: } 2^{nd} \text{ COIN IS HEADS}\end{array}\right\}$ INDEPENDENT EVENTS.

PROBABILITY :

EXPERIMENT THAT CAN BE REPEATED.

DEFINE TWO EVENTS A, B.

EXPERIMENT NUMBER	DID A OCCUR?	DID B OCCUR?
1	✓	×
2	×	✓
3	×	×
4	✓	✓
5	✓	×
⋮	⋮	⋮
n	×	✓
TOTAL	#A	#B

$$P(A) = \frac{\#A}{n} \qquad P(B) = \frac{\#B}{n}$$

CONDITIONAL PROBABILITY : WHAT IS THE PROBABILITY OF A, GIVEN THAT B HAS OCCURRED?

e.g. WHAT IS THE PROB THAT A US CITIZEN IS A REGISTERED DEMOCRAT? 50%

WHAT ...... DEMOCRAT, GIVEN THAT THEY LIVE IN CA? INCREASE TO 60%

WHAT IS THE PROBABILITY OF A, GIVEN THAT B HAS OCCURRED?

$$P(A \mid B)$$

"PROBABILITY OF A GIVEN B"

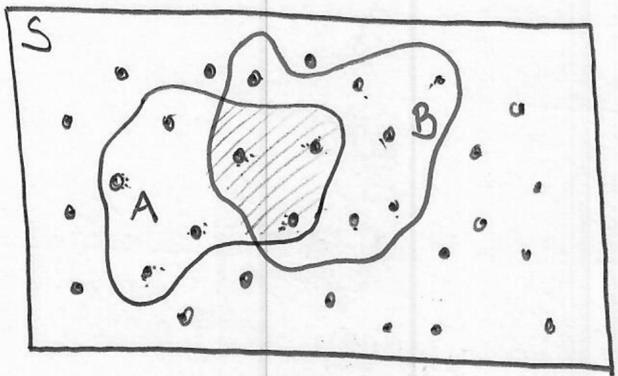
EXPERIMENT TRIAL	A?	B?
~~1~~	✓	
2		✓
3	✓	✓
~~4~~		
5		✓
6	✓	✓
7	✓	✓
8		✓
~~9~~	✓	
10	✓	✓
~~11~~	✓	
⋮	⋮	⋮
n	✓	✓

$$P(A \mid B) = \frac{\#(A \cap B)}{\# B} = \frac{\left(\frac{\#(A \cap B)}{n}\right)}{\left(\frac{\# B}{n}\right)}$$

CONDITIONAL PROBABILITY

$$\boxed{P(A \mid B) = \frac{P(A \cap B)}{P(B)}}$$

IF WE ASSUME ALL SIMPLE EVENTS ARE EQUALLY LIKELY,



GIVEN A STUDENT TAKES GERMAN, WHAT IS PROB. THEY ARE IN CHESS CLUB?

$$\frac{3}{7} \quad \frac{\text{GERMAN + CHESS CLUB}}{\text{GERMAN}}$$

$$= .4286$$

EXPERIMENT: SELECT A STUDENT AT RANDOM (30 STUDENTS)

$$P(A) = \frac{7}{30} = .2333$$

S = STUDENTS IN A CLASSROOM

A = STUDENTS TAKING GERMAN.

B = STUDENTS IN CHESS CLUB.

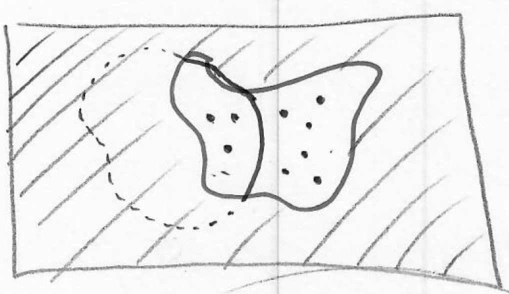
$$P(B) = \frac{9}{30} = .3$$

$$P(A \cap B) = \frac{3}{30} = .1$$

$P(A|B)$ PROBABILITY A STUDENT TAKES GERMAN, GIVEN THAT THEY ARE IN CHESS CLUB.

(PROB. A CHESS CLUB MEMBER TAKES GERMAN)

SHRINKING SAMPLE SPACE TO B.



$$P(A|B) = \frac{3}{9} = .3333$$

FORMULA: $P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{.1}{.3} =$

FOLLOW UP: $P(B|A) = \dfrac{P(B \cap A)}{P(A)} = \dfrac{.1}{.2333} = .4286$

$P(A) = .2333$ PROB STUDENT TAKES GERMAN

$P(A|B) = .3333$ PROB STUDEN TAKES GERMAN,
GIVEN THAT THEY ARE IN CHESS CLUB.

ARE A & B INDEPENDENT? No.

A = GERMAN, B = CHESS CLUB

$P(B) = .3$ PROB STUDENT IS IN CHESS CLUB

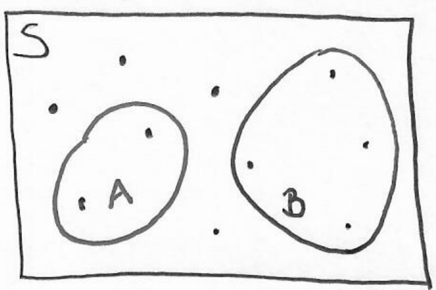
$P(B|A) = .4286$ PROB STUDENT IS IN CHESS CLUB,
GIVEN THAT THEY TAKE GERMAN.

ARE A & B INDEPENDENT? No.

PROPERTIES OF EVENTS A & B: INDEPENDENT, MUTUALLY EXCLUSIVE } DIFFERENT.

(MUTUALLY EXCLUSIVE $\Rightarrow$ NOT INDEPENDENT)

(INDEPENDENT $\Rightarrow$ NOT MUTUALLY EXCLUSIVE).



A & B ARE MUTUALLY EXCLUSIVE.

NOT INDEPENDENT

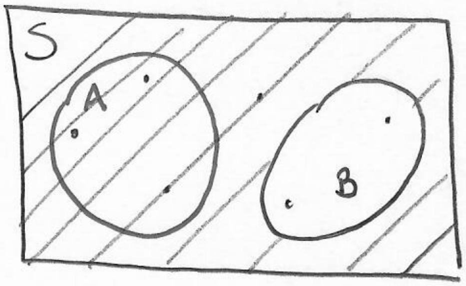
$P(A) = .2$

$P(B) = .4$

$\underbrace{P(A \cap B) = 0}_{\text{MUTUALLY EXCLUSIVE}}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0}{.4} = 0$$

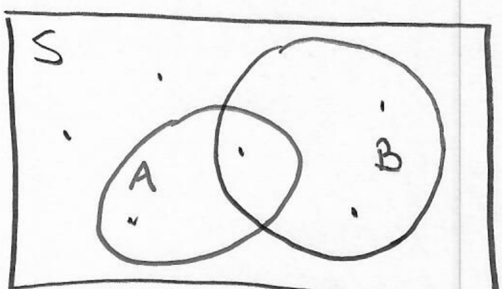
$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0}{.2} = 0$$



$$P(A|B) = \frac{0}{2} = 0$$

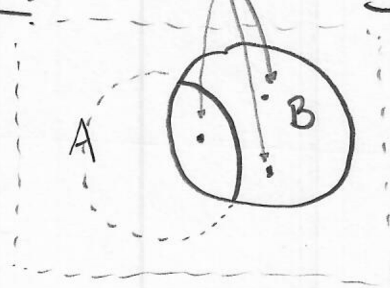
IF A & B ARE MUTUALLY EXCLUSIVE

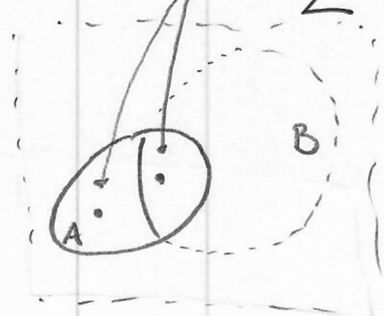
THEN $P(A|B) = P(B|A) = 0$.



$P(A) = \frac{2}{6}$ $P(B) = \frac{3}{6}$

$P(A|B) = \frac{1}{3}$ $P(B|A) = \frac{1}{2}$





ARE A & B INDEPENDENT? YES.

$$P(A) = P(A|B), \quad P(B) = P(B|A)$$

ex. AN EXPERIMENT CAN RESULT IN EVENT A, EVENT B, BOTH A AND B, OR NEITHER A NOR B, WITH THE FOLLOWING PROBABILITIES.

	A	A^c	
B	$P(A \cap B)$ .4	$P(A^c \cap B)$ .3	$.7 = P(B) = P(B \cap A) + P(B \cap A^c)$
B^c	$P(A \cap B^c)$ .2	$P(A^c \cap B^c)$ .1	.3
	.6	.4	1 ✓

$$.6 = P(A) = P(A \cap B) + P(A \cap B^c)$$

ARE A & B MUTUALLY EXCLUSIVE? $P(A \cap B) = 0$? No.

$(P(A \cap B) = .4 \neq 0)$

ARE A & B INDEPENDENT? $P(A) = .6$

DIFFERENT

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{.4}{.7} = .5714$$

No, A & B ARE NOT INDEPENDENT.

SUMMARY: MUTUALLY EXCLUSIVE MEANS $P(A \cap B) = 0$.

INDEPENDENT MEANS $\left.\begin{array}{l} P(A|B) = P(A) \\ P(B|A) = P(B) \end{array}\right\}$ EQUIVALENT

(NOT INDEPENDENT = DEPENDENT)

Note: IF A, B INDEPENDENT THEN $P(A) = P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

DEF OF INDEPENDENCE

DEF. OF COND. PROB.

$$\Rightarrow \quad P(A) = \frac{P(A \cap B)}{P(B)}$$

$$\Rightarrow \quad P(A)P(B) = P(A \cap B)$$

* ONLY TRUE WHEN A & B ARE INDEPENDENT

ex. A, B INDEPENDENT. UNIQUE

$P(A) = .8$

$P(B) = .6$

	A	A^c	
B	.48	.12	.6
B^c	.32	.08	.4
	.8	.2	

A, B NOT INDEPENDENT ∞ MANY WAYS!

$P(A) = .8$

$P(B) = .6$

	A	A^c	
B	.4 / .5	.2 / .1	.6
B^c	.4 / .3	0 / .1	.4
	.8	.2	

MULTIPLICATION RULE:

GIVEN ANY EVENTS A, B, WE HAVE

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad \text{(DEF. OF COND. PROB.)}$$

$$\boxed{P(A)P(B|A) = P(A \cap B)}$$

$$\boxed{P(A \cap B) = P(A)P(B|A)}$$

ex. EXPERIMENT: ROLL A DIE.

IF YOU ROLL 1-4 → PICK A MARBLE FROM BOX 1.

IF YOU ROLL 5-6 → PICK A MARBLE FROM BOX 2.

BOX 1: 3 RED, 8 BLUE.

BOX 2: 7 RED, 1 BLUE.

INTRODUCE NOTATION FOR EVENTS.

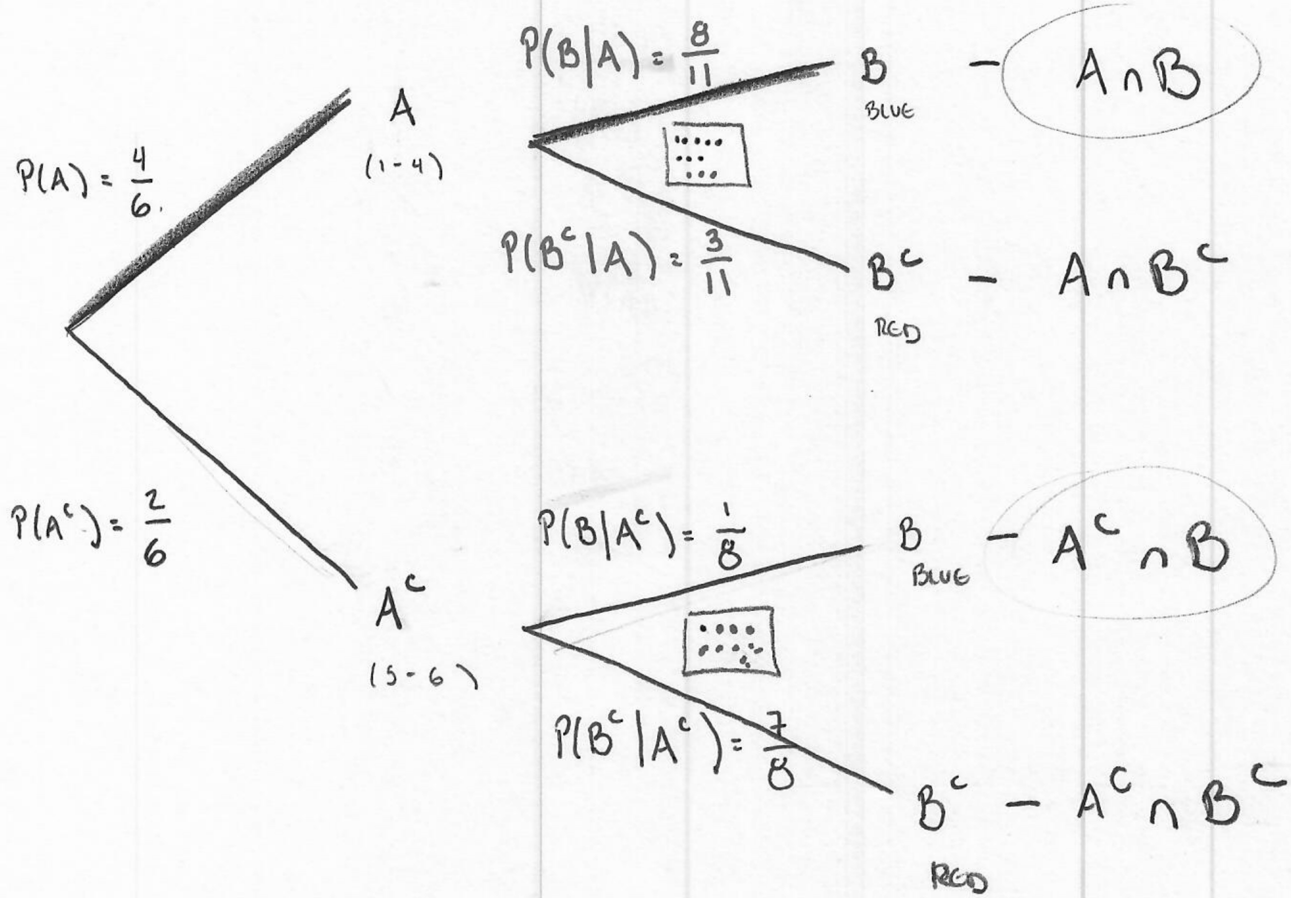
LET A = EVENT THAT YOU ROLL 1-4

A^c = EVENT THAT YOU ROLL 5-6

B = EVENT THAT YOU PICK BLUE MARBLE

B^c = EVENT THAT YOU PICK RED MARBLE

ROLL DIE PICK MARBLE



FIND PROBABILITY YOU $\underbrace{\text{ROLL 1-4}}_{A}$ & $\underbrace{\text{PICK A BLUE MARBLE}}_{B}$.

i.e. FIND $P(A \cap B) = P(A)P(B|A)$

MULTIPLICATION RULE

$$= \left(\frac{4}{6}\right)\left(\frac{8}{11}\right) = \frac{32}{66} = .4848$$

$$P(A^c \cap B) = P(A^c)P(B|A^c) =$$

$$= \left(\frac{2}{6}\right)\left(\frac{1}{8}\right) = \frac{2}{48} = .0417$$

4.29 (a) REPLACING CARDS.

~~- ALL DIFFERENT: ${}_{52}P_3$~~

$$\underset{1^{st}}{\underline{52}} \times \underset{2^{nd}}{\underline{52}} \times \underset{3^{rd}}{\underline{52}} = 140{,}608$$

(b) $${}_{52}P_3 = \underset{1^{st}}{\underline{52}} \times \underset{2^{nd}}{\underline{51}} \times \underset{3^{rd}}{\underline{50}}$$

$$= 132{,}600$$

(c) $$P(\text{SAME CARD}) = \frac{\text{\# WAYS TO ALL GET SAME CARD}}{\text{\# WAYS TO DRAW CARDS IN GENERAL}}$$

$$= \frac{52}{140{,}608} \quad \leftarrow (a)$$

$$= .0003698$$

(d) $$\frac{\text{\# WAYS ALL DIFFERENT}}{\text{\# WAYS IN GENERAL}} = \frac{132{,}600 \leftarrow (b)}{140{,}608 \leftarrow (a)}$$

$$= .9430$$

(c) $\underbrace{\boxed{\underset{1^{st}}{\left(\frac{52}{52}\right)}\underset{2^{nd}}{\left(\frac{3}{51}\right)}\underset{3^{rd}}{\left(\frac{2}{50}\right)}\underset{4^{th}}{\left(\frac{1}{49}\right)}}\underset{5^{th}}{\left(\frac{48}{48}\right)}}_{} = \frac{14976}{311875200}$

$= .000048$

$\times 5 = .00024$

$$\frac{{}_{13}C_1 \times {}_{48}C_1}{{}_{52}C_5} = .00024 \quad \left(\frac{624}{2{,}598{,}960}\right)$$

EVENT: DEAL 4-OF-A-KIND

2 STAGES: