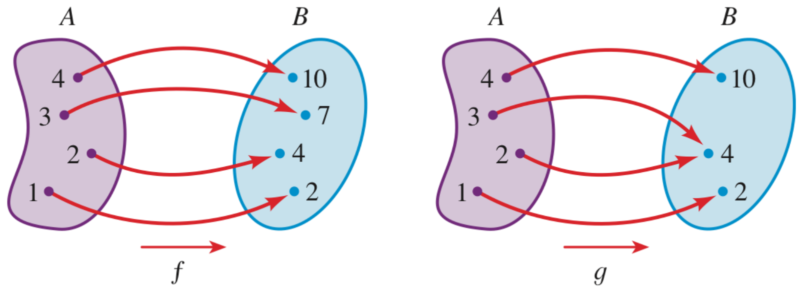
§2.8 One-to-one Functions & Their Inverses



f is one-to-one   *g* is not one-to-one

Observe:

f has a property that

g does not.

Def: A function f is **one-to-one** $(1-1)$ if

different inputs always produce different outputs,

$$a \neq b \Rightarrow f(a) \neq f(b)$$

(i.e. if outputs are equal then inputs are equal,

$$f(a) = f(b) \Rightarrow a = b$$)

Note: f is not $1-1$ if you can find two **different inputs** that produce the **same output.**

e.g. $f(x) = x^2$ is not one-to-one because 2 & -2 are **different inputs** that produce the **same output** $f(2) = f(-2) = 4$.

e.g. $f(x) = 3x + 4$ is one-to-one because

if $f(a) = f(b)$ then $3a + 4 = 3b + 4$

$$3a = 3b$$

$$a = b$$

Don't sweat it. There is another way!

Graphically:

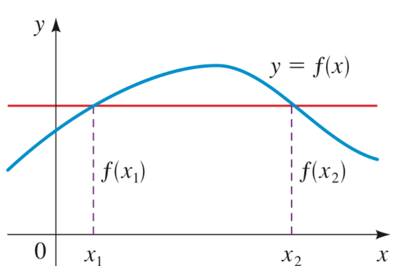
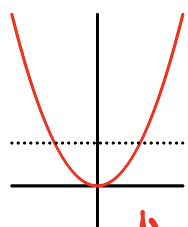


FIGURE 2 This function is not one-to-one because $f(x_1) = f(x_2)$.

HORIZONTAL LINE TEST

A function is one-to-one if and only if no horizontal line intersects its graph more than once.

ex. $y = x^2$



No

ex. $y = 3x + 4$

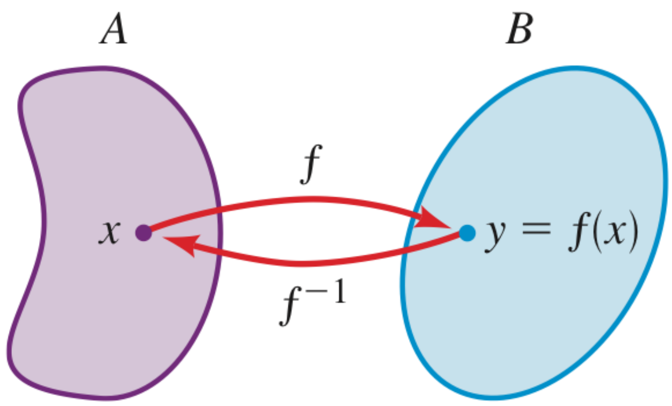
Yes

DEFINITION OF THE INVERSE OF A FUNCTION

Let f be a one-to-one function with domain A and range B. Then its **inverse function** f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$$

for any y in B.



Don't mistake the -1 in f^{-1} for an exponent.

$f^{-1}(x)$ *does not mean* $\dfrac{1}{f(x)}$

The reciprocal $1/f(x)$ is written as $(f(x))^{-1}$.

ex. Suppose f is 1-1 & $f(8) = -3$, $f(-3) = 24$, $f(0) = 8$.

Find $f^{-1}(8)$, $f^{-1}(-3)$, & $f^{-1}(24)$.

USING A TABLE OR GRAPH TO FIND INVERSE VALUES:

x	$h(x)$
2	5
3	8
4	12
5	1
6	3
7	15

Finding values of h^{-1} from a table of h

$h^{-1}(8) = 3$
$h^{-1}(12) = 4$
$h^{-1}(3) = 6$

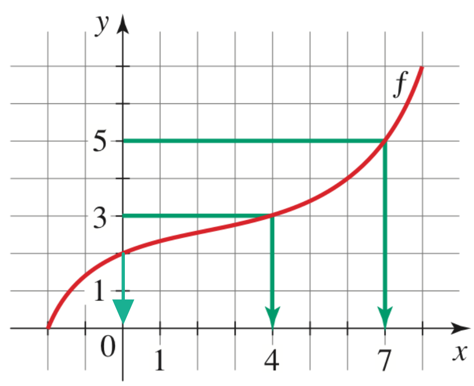


FIGURE 8 Finding values of f^{-1} from a graph of f

$f^{-1}(5) = 7$
$f^{-1}(3) = 4$
$f^{-1}(2) = 0$

* IMPORTANT *

INVERSE FUNCTION PROPERTY

Let f be a one-to-one function with domain A and range B. The inverse function f^{-1} satisfies the following cancellation properties:

$$f^{-1}(f(x)) = x \quad \text{for every } x \text{ in } A$$

$$f(f^{-1}(x)) = x \quad \text{for every } x \text{ in } B$$

Conversely, any function f^{-1} satisfying these equations is the inverse of f.

ex. IS IT TRUE THAT $f(x) = 4x+3$ & $g(x) = \dfrac{x-3}{4}$

ARE INVERSES OF EACHOTHER ?

HOW TO FIND THE INVERSE OF A ONE-TO-ONE FUNCTION

1. Write $y = f(x)$.
2. Solve this equation for x in terms of y (if possible).
3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.

ex. GIVEN $f(x) = \dfrac{5x-7}{4}$, FIND $f^{-1}(x)$.

ex. GIVEN $f(x) = \dfrac{2}{1-x}$, FIND $f^{-1}(x)$.

ex. GIVEN $f(x) = \dfrac{x+1}{x-1}$, FIND $f^{-1}(x)$.

TRICKIER: FIRST IDENTIFY DOMAIN & RANGE OF f

(i.e. RANGE & DOMAIN OF f^{-1})

ex. $f(x) = x^2 - 4$, $x \geq 0$

ex. $f(x) = 2 + \sqrt{x-3}$

GRAPHS OF INVERSE FUNCTIONS

SUPPOSE f IS 1-1.

IF (a, b) IS ON THE GRAPH $y = f(x)$
THEN (b, a) IS ON THE GRAPH $y = f^{-1}(x)$

The graph of f^{-1} is obtained by reflecting the graph of f in the line $y = x$.

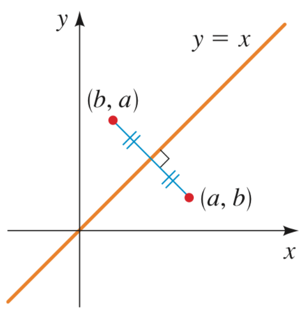


FIGURE 9

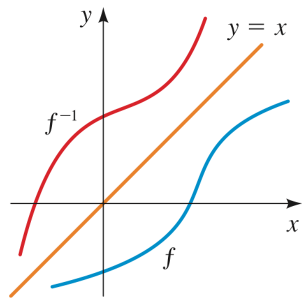


FIGURE 10

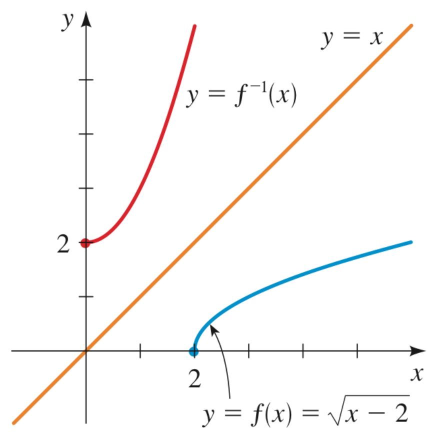
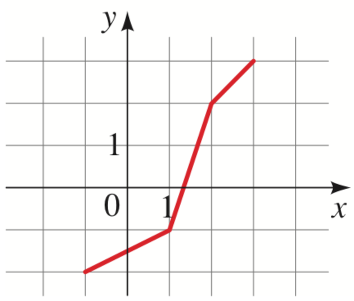


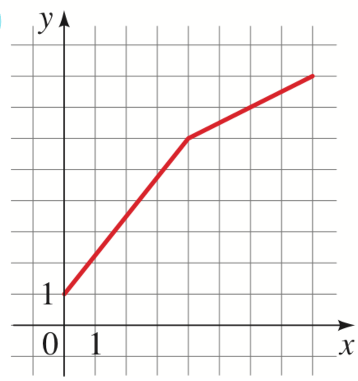
FIGURE 11

89–90 ■ Graph of an Inverse Function Use the graph of f to sketch the graph of f^{-1}.

89.



90.



ADDITIONAL EXERCISES:

61. $f(x) = 4 - x^2, \quad x \geq 0$

62. $f(x) = x^2 + x, \quad x \geq -\frac{1}{2}$

63. $f(x) = x^6, \quad x \geq 0$

64. $f(x) = \dfrac{1}{x^2}, \quad x > 0$

65. $f(x) = \dfrac{2 - x^3}{5}$

66. $f(x) = (x^5 - 6)^7$

67. $f(x) = \sqrt{5 + 8x}$

68. $f(x) = 2 + \sqrt{3 + x}$

69. $f(x) = 2 + \sqrt[3]{x}$

70. $f(x) = \sqrt{4 - x^2}, \quad 0 \leq x \leq 2$