61–62 ■ Present Value The **present value** of a sum of money is the amount that must be invested now, at a given rate of interest, to produce the desired sum at a later date.

61. Find the present value of \$10,000 if interest is paid at a rate of 9% per year, compounded semiannually, for 3 years.

62. Find the present value of \$100,000 if interest is paid at a rate of 8% per year, compounded monthly, for 5 years.

63. Annual Percentage Yield Find the annual percentage yield for an investment that earns 8% per year, compounded monthly.

64. Annual Percentage Yield Find the annual percentage yield for an investment that earns $5\frac{1}{2}\%$ per year, compounded quarterly.

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

65. DISCUSS ■ DISCOVER: Growth of an Exponential Function Suppose you are offered a job that lasts one month, and you are to be very well paid. Which of the following methods of payment is more profitable for you?

(a) One million dollars at the end of the month

(b) Two cents on the first day of the month, 4 cents on the second day, 8 cents on the third day, and, in general, 2^n cents on the nth day

66. DISCUSS ■ DISCOVER: The Height of the Graph of an Exponential Function Your mathematics instructor asks you to sketch a graph of the exponential function

$$f(x) = 2^x$$

for x between 0 and 40, using a scale of 10 units to one inch. What are the dimensions of the sheet of paper you will need to sketch this graph?

4.2 THE NATURAL EXPONENTIAL FUNCTION

■ The Number e ■ The Natural Exponential Function ■ Continuously Compounded Interest

Any positive number can be used as a base for an exponential function. In this section we study the special base e, which is convenient for applications involving calculus.

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The **Gateway Arch** in St. Louis, Missouri, is shaped in the form of the graph of a combination of exponential functions (*not* a parabola, as it might first appear). Specifically, it is a **catenary**, which is the graph of an equation of the form

$$y = a(e^{bx} + e^{-bx})$$

(see Exercises 17 and 19). This shape was chosen because it is optimal for distributing the internal structural forces of the arch. Chains and cables suspended between two points (for example, the stretches of cable between pairs of telephone poles) hang in the shape of a catenary.

■ The Number e

The number e is defined as the value that $(1 + 1/n)^n$ approaches as n becomes large. (In calculus this idea is made more precise through the concept of a limit.) The table shows the values of the expression $(1 + 1/n)^n$ for increasingly large values of n.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2.00000
5	2.48832
10	2.59374
100	2.70481
1000	2.71692
10,000	2.71815
100,000	2.71827
1,000,000	2.71828

It appears that, rounded to five decimal places, $e \approx 2.71828$; in fact, the approximate value to 20 decimal places is

$$e \approx 2.71828182845904523536$$

It can be shown that e is an irrational number, so we cannot write its exact value in decimal form.

■ The Natural Exponential Function

The notation e was chosen by Leonhard Euler (see page 63), probably because it is the first letter of the word *exponential*.

The number e is the base for the natural exponential function. Why use such a strange base for an exponential function? It might seem at first that a base such as 10 is easier to work with. We will see, however, that in certain applications the number e is the best

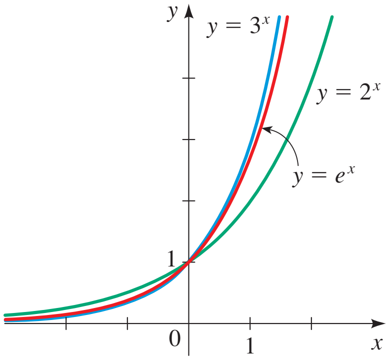


FIGURE 1 Graph of the natural exponential function

possible base. In this section we study how e occurs in the description of compound interest.

THE NATURAL EXPONENTIAL FUNCTION

The **natural exponential function** is the exponential function

$$f(x) = e^x$$

with base e. It is often referred to as *the* exponential function.

Since $2 < e < 3$, the graph of the natural exponential function lies between the graphs of $y = 2^x$ and $y = 3^x$, as shown in Figure 1.

Scientific calculators have a special key for the function $f(x) = e^x$. We use this key in the next example.

EXAMPLE 1 ■ Evaluating the Exponential Function

Evaluate each expression rounded to five decimal places.

(a) e^3 **(b)** $2e^{-0.53}$ **(c)** $e^{4.8}$

SOLUTION We use the e^x key on a calculator to evaluate the exponential function.

(a) $e^3 \approx 20.08554$ **(b)** $2e^{-0.53} \approx 1.17721$ **(c)** $e^{4.8} \approx 121.51042$

Now Try Exercise 3 ■

EXAMPLE 2 ■ Graphing the Exponential Functions

Sketch the graph of each function. State the domain, range, and asymptote.

(a) $f(x) = e^{-x}$ **(b)** $g(x) = 3e^{0.5x}$

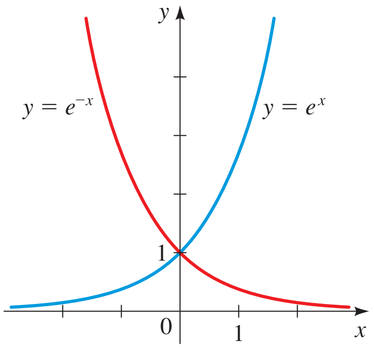


FIGURE 2

SOLUTION

(a) We start with the graph of $y = e^x$ and reflect in the y-axis to obtain the graph of $y = e^{-x}$ as in Figure 2. From the graph we see that the domain of f is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

(b) We calculate several values, plot the resulting points, then connect the points with a smooth curve. The graph is shown in Figure 3. From the graph we see that the domain of g is the set $\mathbb{R}$ of all real numbers, the range is the interval $(0, \infty)$, and the line $y = 0$ is a horizontal asymptote.

x	$f(x) = 3e^{0.5x}$
−3	0.67
−2	1.10
−1	1.82
0	3.00
1	4.95
2	8.15
3	13.45

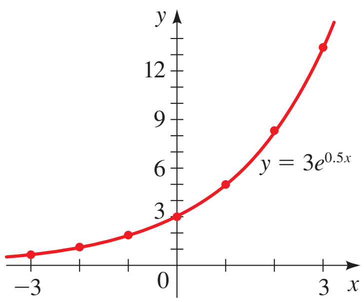


FIGURE 3

Now Try Exercises 5 and 7 ■

EXAMPLE 3 ■ An Exponential Model for the Spread of a Virus

An infectious disease begins to spread in a small city of population 10,000. After t days, the number of people who have succumbed to the virus is modeled by the function

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

(a) How many infected people are there initially (at time $t = 0$)?

(b) Find the number of infected people after one day, two days, and five days.

(c) Graph the function v, and describe its behavior.

SOLUTION

(a) Since $v(0) = 10{,}000/(5 + 1245e^{0}) = 10{,}000/1250 = 8$, we conclude that 8 people initially have the disease.

(b) Using a calculator, we evaluate $v(1)$, $v(2)$, and $v(5)$ and then round off to obtain the following values.

Days	Infected people
1	21
2	54
5	678

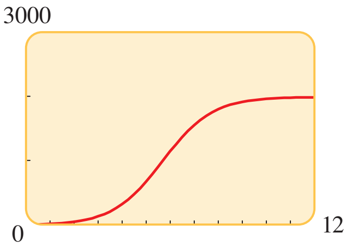


FIGURE 4

$$v(t) = \frac{10{,}000}{5 + 1245e^{-0.97t}}$$

(c) From the graph in Figure 4 we see that the number of infected people first rises slowly, then rises quickly between day 3 and day 8, and then levels off when about 2000 people are infected.

Now Try Exercise 27 ■

The graph in Figure 4 is called a *logistic curve* or a *logistic growth model*. Curves like it occur frequently in the study of population growth. (See Exercises 27–30.)

■ Continuously Compounded Interest

In Example 6 of Section 4.1 we saw that the interest paid increases as the number of compounding periods n increases. Let's see what happens as n increases indefinitely. If we let $m = n/r$, then

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt} = P\left[\left(1 + \frac{r}{n}\right)^{n/r}\right]^{rt} = P\left[\left(1 + \frac{1}{m}\right)^{m}\right]^{rt}$$

Recall that as m becomes large, the quantity $(1 + 1/m)^m$ approaches the number e. Thus the amount approaches $A = Pe^{rt}$. This expression gives the amount when the interest is compounded at "every instant."

CONTINUOUSLY COMPOUNDED INTEREST

Continuously compounded interest is calculated by the formula

$$A(t) = Pe^{rt}$$

where $A(t)$ = amount after t years
P = principal
r = interest rate per year
t = number of years

EXAMPLE 4 ■ Calculating Continuously Compounded Interest

Find the amount after 3 years if \$1000 is invested at an interest rate of 12% per year, compounded continuously.

SOLUTION We use the formula for continuously compounded interest with $P = \$1000$, $r = 0.12$, and $t = 3$ to get

$$A(3) = 1000e^{(0.12)3} = 1000e^{0.36} = \$1433.33$$

Compare this amount with the amounts in Example 6 of Section 4.1.

Now Try Exercise 33 ■

4.2 EXERCISES

CONCEPTS

1. The function $f(x) = e^x$ is called the ________ exponential function. The number e is approximately equal to ________.
2. In the formula $A(t) = Pe^{rt}$ for continuously compound interest, the letters P, r, and t stand for ________, ________, and ________, respectively, and $A(t)$ stands for ________. So if \$100 is invested at an interest rate of 6% compounded continuously, then the amount after 2 years is ________.

SKILLS

3–4 ■ Evaluating Exponential Functions Use a calculator to evaluate the function at the indicated values. Round your answers to three decimals.

3. $h(x) = e^x$; $h(1), h(\pi), h(-3), h(\sqrt{2})$
4. $h(x) = e^{-3x}$; $h(\frac{1}{3}), h(1.5), h(-1), h(-\pi)$

5–6 ■ Graphing Exponential Functions Complete the table of values, rounded to two decimal places, and sketch a graph of the function.

5.

x	$f(x) = 1.5e^x$
-2	
-1	
-0.5	
0	
0.5	
1	
2	

6.

x	$f(x) = 4e^{-x/3}$
-3	
-2	
-1	
0	
1	
2	
3	

7–16 ■ Graphing Exponential Functions Graph the function, not by plotting points, but by starting from the graph of $y = e^x$ in Figure 1. State the domain, range, and asymptote.

7. $g(x) = 2 + e^x$
8. $h(x) = e^{-x} - 3$
9. $f(x) = -e^x$
10. $y = 1 - e^x$
11. $y = e^{-x} - 1$
12. $f(x) = -e^{-x}$
13. $f(x) = e^{x-2}$
14. $y = e^{x-3} + 4$
15. $h(x) = e^{x+1} - 3$
16. $g(x) = -e^{x-1} - 2$

SKILLS Plus

17. **Hyperbolic Cosine Function** The *hyperbolic cosine function* is defined by

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

(a) Sketch the graphs of the functions $y = \frac{1}{2}e^x$ and $y = \frac{1}{2}e^{-x}$ on the same axes, and use graphical addition (see Section 2.7) to sketch the graph of $y = \cosh(x)$.

(b) Use the definition to show that $\cosh(-x) = \cosh(x)$.

18. **Hyperbolic Sine Function** The *hyperbolic sine function* is defined by

$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

(a) Sketch the graph of this function using graphical addition as in Exercise 17.

(b) Use the definition to show that $\sinh(-x) = -\sinh(x)$

19. **Families of Functions**

(a) Draw the graphs of the family of functions

$$f(x) = \frac{a}{2}(e^{x/a} + e^{-x/a})$$

for $a = 0.5$, 1, 1.5, and 2.

(b) How does a larger value of a affect the graph?

20. ***The Definition of e*** Illustrate the definition of the number e by graphing the curve $y = (1 + 1/x)^x$ and the line $y = e$ on the same screen, using the viewing rectangle $[0, 40]$ by $[0, 4]$.

21–22 ■ Local Extrema Find the local maximum and minimum values of the function and the value of x at which each occurs. State each answer rounded to two decimal places.

21. $g(x) = x^x$, $x > 0$
22. $g(x) = e^x + e^{-2x}$

APPLICATIONS

23. Medical Drugs When a certain medical drug is administered to a patient, the number of milligrams remaining in the patient's bloodstream after t hours is modeled by

$$D(t) = 50e^{-0.2t}$$

How many milligrams of the drug remain in the patient's bloodstream after 3 hours?

24. Radioactive Decay A radioactive substance decays in such a way that the amount of mass remaining after t days is given by the function

$$m(t) = 13e^{-0.015t}$$

where $m(t)$ is measured in kilograms.

(a) Find the mass at time $t = 0$.

(b) How much of the mass remains after 45 days?

25. Sky Diving A sky diver jumps from a reasonable height above the ground. The air resistance she experiences is proportional to her velocity, and the constant of proportionality is 0.2. It can be shown that the downward velocity of the sky diver at time t is given by

$$v(t) = 180(1 - e^{-0.2t})$$

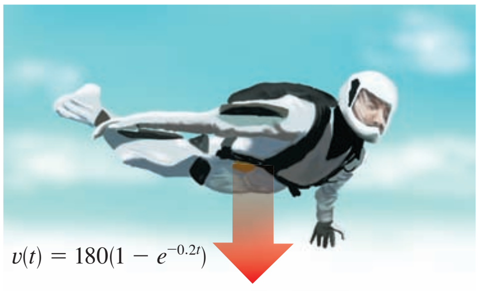
where t is measured in seconds (s) and $v(t)$ is measured in feet per second (ft/s).

(a) Find the initial velocity of the sky diver.

(b) Find the velocity after 5 s and after 10 s.

(c) Draw a graph of the velocity function $v(t)$.

(d) The maximum velocity of a falling object with wind resistance is called its *terminal velocity*. From the graph in part (c) find the terminal velocity of this sky diver.



26. Mixtures and Concentrations A 50-gal barrel is filled completely with pure water. Salt water with a concentration of 0.3 lb/gal is then pumped into the barrel, and the resulting mixture overflows at the same rate. The amount of salt in the barrel at time t is given by

$$Q(t) = 15(1 - e^{-0.04t})$$

where t is measured in minutes and $Q(t)$ is measured in pounds.

(a) How much salt is in the barrel after 5 min?

(b) How much salt is in the barrel after 10 min?

(c) Draw a graph of the function $Q(t)$.

(d) Use the graph in part (c) to determine the value that the amount of salt in the barrel approaches as t becomes large. Is this what you would expect?

$Q(t) = 15(1 - e^{-0.04t})$

27. Logistic Growth Animal populations are not capable of unrestricted growth because of limited habitat and food supplies. Under such conditions the population follows a *logistic growth model*:

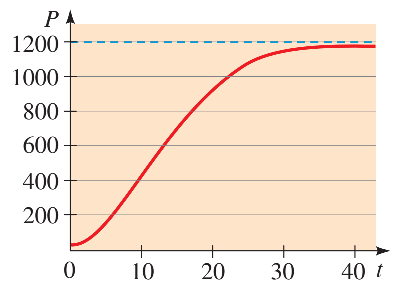
$$P(t) = \frac{d}{1 + ke^{-ct}}$$

where c, d, and k are positive constants. For a certain fish population in a small pond $d = 1200$, $k = 11$, $c = 0.2$, and t is measured in years. The fish were introduced into the pond at time $t = 0$.

(a) How many fish were originally put in the pond?

(b) Find the population after 10, 20, and 30 years.

(c) Evaluate $P(t)$ for large values of t. What value does the population approach as $t \to \infty$? Does the graph shown confirm your calculations?



28. Bird Population The population of a certain species of bird is limited by the type of habitat required for nesting. The population behaves according to the logistic growth model

$$n(t) = \frac{5600}{0.5 + 27.5e^{-0.044t}}$$

where t is measured in years.

(a) Find the initial bird population.

(b) Draw a graph of the function $n(t)$.

(c) What size does the population approach as time goes on?

29. World Population The relative growth rate of world population has been decreasing steadily in recent years. On the basis of this, some population models predict that world population will eventually stabilize at a level that the planet can support. One such logistic model is

$$P(t) = \frac{73.2}{6.1 + 5.9e^{-0.02t}}$$

where $t = 0$ is the year 2000 and population is measured in billions.

(a) What world population does this model predict for the year 2200? For 2300?

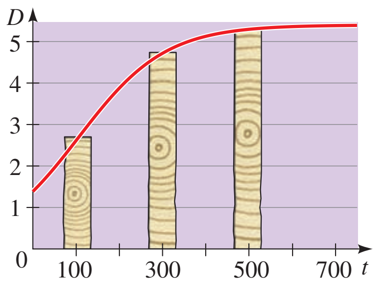
(b) Sketch a graph of the function P for the years 2000 to 2500.

(c) According to this model, what size does the world population seem to approach as time goes on?

30. Tree Diameter For a certain type of tree the diameter D (in feet) depends on the tree's age t (in years) according to the logistic growth model

$$D(t) = \frac{5.4}{1 + 2.9e^{-0.01t}}$$

Find the diameter of a 20-year-old tree.



31–32 ■ Compound Interest An investment of $7000 is deposited into an account in which interest is compounded continuously. Complete the table by filling in the amounts to which the investment grows at the indicated times or interest rates.

31. $r = 3\%$

Time (years)	Amount
1	
2	
3	
4	
5	
6	

32. $t = 10$ years

Rate per year	Amount
1%	
2%	
3%	
4%	
5%	
6%	

33. Compound Interest If $2000 is invested at an interest rate of 3.5% per year, compounded continuously, find the value of the investment after the given number of years.

(a) 2 years

(b) 4 years

(c) 12 years

34. Compound Interest If $3500 is invested at an interest rate of 6.25% per year, compounded continuously, find the value of the investment after the given number of years.

(a) 3 years

(b) 6 years

(c) 9 years

35. Compound Interest If $600 is invested at an interest rate of 2.5% per year, find the amount of the investment at the end of 10 years for the following compounding methods.

(a) Annually

(b) Semiannually

(c) Quarterly

(d) Continuously

36. Compound Interest If $8000 is invested in an account for which interest is compounded continuously, find the amount of the investment at the end of 12 years for the following interest rates.

(a) 2%

(b) 3%

(c) 4.5%

(d) 7%

37. Compound Interest Which of the given interest rates and compounding periods would provide the best investment?

(a) $2\frac{1}{2}\%$ per year, compounded semiannually

(b) $2\frac{1}{4}\%$ per year, compounded monthly

(c) 2% per year, compounded continuously

38. Compound Interest Which of the given interest rates and compounding periods would provide the better investment?

(a) $5\frac{1}{8}\%$ per year, compounded semiannually

(b) 5% per year, compounded continuously

39. Investment A sum of $5000 is invested at an interest rate of 9% per year, compounded continuously.

(a) Find the value $A(t)$ of the investment after t years.

(b) Draw a graph of $A(t)$.

(c) Use the graph of $A(t)$ to determine when this investment will amount to $25,000.