4.3 LOGARITHMIC FUNCTIONS

■ Logarithmic Functions ■ Graphs of Logarithmic Functions ■ Common Logarithms ■ Natural Logarithms

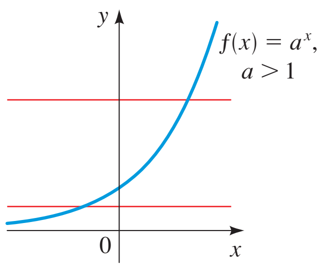

FIGURE 1 $f(x) = a^x$ is one-to-one.

In this section we study the inverses of exponential functions.

■ Logarithmic Functions

Every exponential function $f(x) = a^x$, with $a > 0$ and $a \neq 1$, is a one-to-one function by the Horizontal Line Test (see Figure 1 for the case $a > 1$) and therefore has an inverse function. The inverse function f^{-1} is called the *logarithmic function with base a* and is denoted by $\log_a$. Recall from Section 2.8 that f^{-1} is defined by

$$f^{-1}(x) = y \quad \Leftrightarrow \quad f(y) = x$$

This leads to the following definition of the logarithmic function.

We read $\log_a x = y$ as "log base a of x is y."

DEFINITION OF THE LOGARITHMIC FUNCTION

Let a be a positive number with $a \neq 1$. The **logarithmic function with base a**, denoted by $\mathbf{log}_a$, is defined by

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

So $\log_a x$ is the *exponent* to which the base a must be raised to give x.

By tradition the name of the logarithmic function is $\log_a$, not just a single letter. Also, we usually omit the parentheses in the function notation and write

$$\log_a(x) = \log_a x$$

When we use the definition of logarithms to switch back and forth between the **logarithmic form** $\log_a x = y$ and the **exponential form** $a^y = x$, it is helpful to notice that, in both forms, the base is the same.

Logarithmic form	Exponential form
Exponent	Exponent
$\log_a x = y$	$a^y = x$
Base	Base

EXAMPLE 1 ■ Logarithmic and Exponential Forms

The logarithmic and exponential forms are equivalent equations: If one is true, then so is the other. So we can switch from one form to the other as in the following illustrations.

Logarithmic form	Exponential form
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_2 8 = 3$	$2^3 = 8$
$\log_2\left(\frac{1}{8}\right) = -3$	$2^{-3} = \frac{1}{8}$
$\log_5 s = r$	$5^r = s$

Now Try Exercise 7 ■

x	$\log_{10} x$
10^4	4
10^3	3
10^2	2
10	1
1	0
10^{-1}	-1
10^{-2}	-2
10^{-3}	-3
10^{-4}	-4

It is important to understand that $\log_a x$ is an *exponent*. For example, the numbers in the right-hand column of the table in the margin are the logarithms (base 10) of the numbers in the left-hand column. This is the case for all bases, as the following example illustrates.

EXAMPLE 2 ■ Evaluating Logarithms

(a) $\log_{10} 1000 = 3$ because $10^3 = 1000$

(b) $\log_2 32 = 5$ because $2^5 = 32$

(c) $\log_{10} 0.1 = -1$ because $10^{-1} = 0.1$

(d) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$

Now Try Exercises 9 and 11 ■

Inverse Function Property:

$f^{-1}(f(x)) = x$

$f(f^{-1}(x)) = x$

When we apply the Inverse Function Property described on page 222 to $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, we get

$$\log_a(a^x) = x \qquad x \in \mathbb{R}$$

$$a^{\log_a x} = x \qquad x > 0$$

We list these and other properties of logarithms discussed in this section.

PROPERTIES OF LOGARITHMS

Property	Reason
1. $\log_a 1 = 0$	We must raise a to the power 0 to get 1.
2. $\log_a a = 1$	We must raise a to the power 1 to get a.
3. $\log_a a^x = x$	We must raise a to the power x to get a^x.
4. $a^{\log_a x} = x$	$\log_a x$ is the power to which a must be raised to get x.

EXAMPLE 3 ■ Applying Properties of Logarithms

We illustrate the properties of logarithms when the base is 5.

$\log_5 1 = 0$ Property 1 $\qquad \log_5 5 = 1$ Property 2

$\log_5 5^8 = 8$ Property 3 $\qquad 5^{\log_5 12} = 12$ Property 4

Now Try Exercises 25 and 31 ■

■ Graphs of Logarithmic Functions

Recall that if a one-to-one function f has domain A and range B, then its inverse function f^{-1} has domain B and range A. Since the exponential function $f(x) = a^x$ with $a \neq 1$ has domain $\mathbb{R}$ and range $(0, \infty)$, we conclude that its inverse function, $f^{-1}(x) = \log_a x$, has domain $(0, \infty)$ and range $\mathbb{R}$.

The graph of $f^{-1}(x) = \log_a x$ is obtained by reflecting the graph of $f(x) = a^x$ in the line $y = x$. Figure 2 shows the case $a > 1$. The fact that $y = a^x$ (for $a > 1$) is a very rapidly increasing function for $x > 0$ implies that $y = \log_a x$ is a very slowly increasing function for $x > 1$ (see Exercise 102).

Since $\log_a 1 = 0$, the x-intercept of the function $y = \log_a x$ is 1. The y-axis is a vertical asymptote of $y = \log_a x$ because $\log_a x \to -\infty$ as $x \to 0^+$.

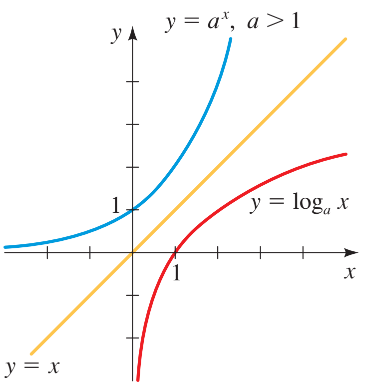

FIGURE 2 Graph of the logarithmic function $f(x) = \log_a x$

EXAMPLE 4 ■ Graphing a Logarithmic Function by Plotting Points

Sketch the graph of $f(x) = \log_2 x$.

SOLUTION To make a table of values, we choose the x-values to be powers of 2 so that we can easily find their logarithms. We plot these points and connect them with a smooth curve as in Figure 3.

x	$\log_2 x$
2^3	3
2^2	2
2	1
1	0
2^{-1}	-1
2^{-2}	-2
2^{-3}	-3
2^{-4}	-4

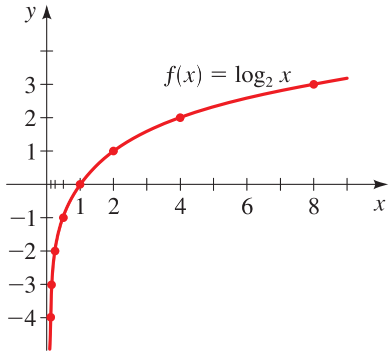

FIGURE 3

Now Try Exercise 49 ■

Figure 4 shows the graphs of the family of logarithmic functions with bases 2, 3, 5, and 10. These graphs are drawn by reflecting the graphs of $y = 2^x$, $y = 3^x$, $y = 5^x$, and $y = 10^x$ (see Figure 2 in Section 4.1) in the line $y = x$. We can also plot points as an aid to sketching these graphs, as illustrated in Example 4.

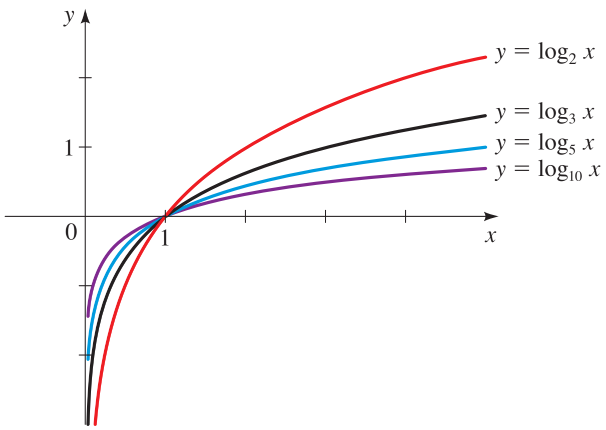

FIGURE 4 A family of logarithmic functions

In the next two examples we graph logarithmic functions by starting with the basic graphs in Figure 4 and using the transformations of Section 2.6.

EXAMPLE 5 ■ Reflecting Graphs of Logarithmic Functions

Sketch the graph of each function. State the domain, range, and asymptote.

(a) $g(x) = -\log_2 x$ **(b)** $h(x) = \log_2(-x)$

SOLUTION

(a) We start with the graph of $f(x) = \log_2 x$ and reflect in the x-axis to get the graph of $g(x) = -\log_2 x$ in Figure 5(a). From the graph we see that the domain of g is $(0, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

Mathematics in the Modern World

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Law Enforcement

Mathematics aids law enforcement in numerous and surprising ways, from the reconstruction of bullet trajectories to determining the time of death to calculating the probability that a DNA sample is from a particular person. One interesting use is in the search for missing persons. A person who has been missing for several years might look quite different from his or her most recent available photograph. This is particularly true if the missing person is a child. Have you ever wondered what you will look like 5, 10, or 15 years from now?

Researchers have found that different parts of the body grow at different rates. For example, you have no doubt noticed that a baby's head is much larger relative to its body than an adult's. As another example, the ratio of arm length to height is $\frac{1}{3}$ in a child but about $\frac{2}{5}$ in an adult. By collecting data and analyzing the graphs, researchers are able to determine the functions that model growth. As in all growth phenomena, exponential and logarithmic functions play a crucial role. For instance, the formula that relates arm length l to height h is $l = ae^{kh}$ where a and k are constants. By studying various physical characteristics of a person, mathematical biologists model each characteristic by a function that describes how it changes over time. Models of facial characteristics can be programmed into a computer to give a picture of how a person's appearance changes over time. These pictures aid law enforcement agencies in locating missing persons.

(b) We start with the graph of $f(x) = \log_2 x$ and reflect in the y-axis to get the graph of $h(x) = \log_2(-x)$ in Figure 5(b). From the graph we see that the domain of h is $(-\infty, 0)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

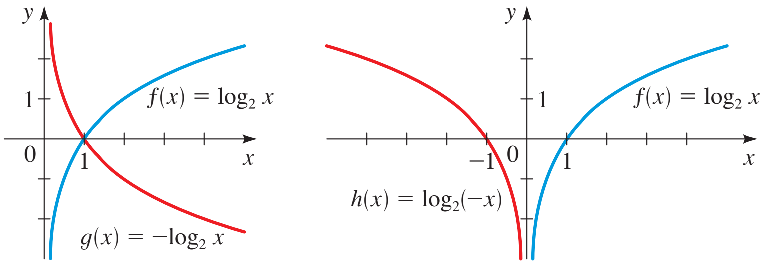

FIGURE 5 (a) (b)

Now Try Exercise 61 ■

EXAMPLE 6 ■ Shifting Graphs of Logarithmic Functions

Sketch the graph of each function. State the domain, range, and asymptote.

(a) $g(x) = 2 + \log_5 x$ **(b)** $h(x) = \log_{10}(x - 3)$

SOLUTION

(a) The graph of g is obtained from the graph of $f(x) = \log_5 x$ (Figure 4) by shifting upward 2 units, as shown in Figure 6. From the graph we see that the domain of g is $(0, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 0$ is a vertical asymptote.

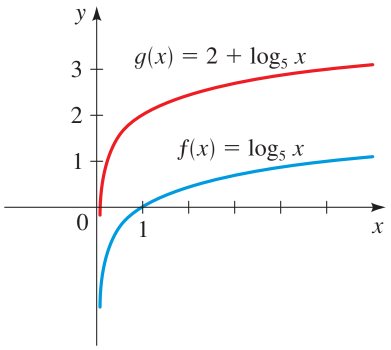

FIGURE 6

(b) The graph of h is obtained from the graph of $f(x) = \log_{10} x$ (Figure 4) by shifting to the right 3 units, as shown in Figure 7. From the graph we see that the domain of h is $(3, \infty)$, the range is the set $\mathbb{R}$ of all real numbers, and the line $x = 3$ is a vertical asymptote.

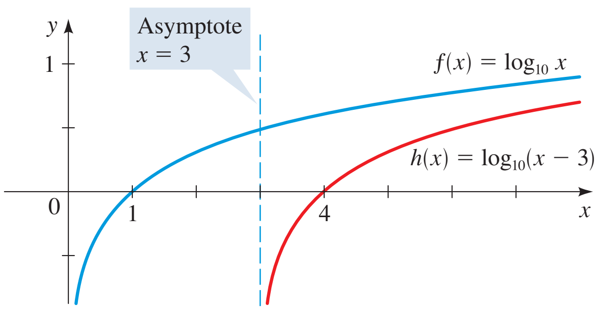

FIGURE 7

Now Try Exercises 63 and 67 ■

Common Logarithms

We now study logarithms with base 10.

COMMON LOGARITHM

The logarithm with base 10 is called the **common logarithm** and is denoted by omitting the base:

$$\log x = \log_{10} x$$

From the definition of logarithms we can easily find that

$$\log 10 = 1 \qquad \text{and} \qquad \log 100 = 2$$

But how do we find log 50? We need to find the exponent y such that $10^y = 50$. Clearly, 1 is too small and 2 is too large. So

$$1 < \log 50 < 2$$

To get a better approximation, we can experiment to find a power of 10 closer to 50. Fortunately, scientific calculators are equipped with a LOG key that directly gives values of common logarithms.

EXAMPLE 7 ■ Evaluating Common Logarithms

Use a calculator to find appropriate values of $f(x) = \log x$, and use the values to sketch the graph.

SOLUTION We make a table of values, using a calculator to evaluate the function at those values of x that are not powers of 10. We plot those points and connect them by a smooth curve as in Figure 8.

x	$\log x$
0.01	−2
0.1	−1
0.5	−0.301
1	0
4	0.602
5	0.699
10	1

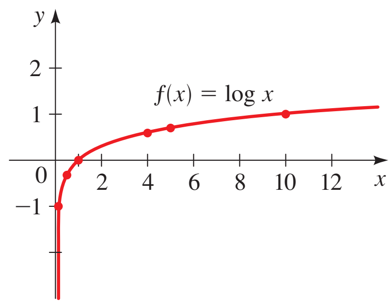

FIGURE 8

Now Try Exercise 51 ■

Scientists model human response to stimuli (such as sound, light, or pressure) using logarithmic functions. For example, the intensity of a sound must be increased manyfold before we "feel" that the loudness has simply doubled. The psychologist Gustav Fechner formulated the law as

$$S = k \log\left(\frac{I}{I_0}\right)$$

where S is the subjective intensity of the stimulus, I is the physical intensity of the stimulus, I_0 stands for the threshold physical intensity, and k is a constant that is different for each sensory stimulus.

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JOHN NAPIER (1550–1617) was a Scottish landowner for whom mathematics was a hobby. We know him today because of his key invention: logarithms, which he published in 1614 under the title *A Description of the Marvelous Rule of Logarithms*. In Napier's time, logarithms were used exclusively for simplifying complicated calculations. For example, to multiply two large numbers, we would write them as powers of 10. The exponents are simply the logarithms of the numbers. For instance,

$$\begin{aligned} 4532 \times 57783 &\approx 10^{3.65629} \times 10^{4.76180} \\ &= 10^{8.41809} \\ &\approx 261{,}872{,}564 \end{aligned}$$

The idea is that multiplying powers of 10 is easy (we simply add their exponents). Napier produced extensive tables giving the logarithms (or exponents) of numbers. Since the advent of calculators and computers, logarithms are no longer used for this purpose. The logarithmic functions, however, have found many applications, some of which are described in this chapter.

Napier wrote on many topics. One of his most colorful works is a book entitled *A Plaine Discovery of the Whole Revelation of Saint John*, in which he predicted that the world would end in the year 1700.

Human response to sound and light intensity is logarithmic.

EXAMPLE 8 ■ Common Logarithms and Sound

We study the decibel scale in more detail in Section 4.7.

The perception of the loudness B (in decibels, dB) of a sound with physical intensity I (in W/m^2) is given by

$$B = 10 \log\left(\frac{I}{I_0}\right)$$

where I_0 is the physical intensity of a barely audible sound. Find the decibel level (loudness) of a sound whose physical intensity I is 100 times that of I_0.

SOLUTION We find the decibel level B by using the fact that $I = 100I_0$.

$$\begin{aligned} B &= 10 \log\left(\frac{I}{I_0}\right) && \text{Definition of } B \\ &= 10 \log\left(\frac{100I_0}{I_0}\right) && I = 100I_0 \\ &= 10 \log 100 && \text{Cancel } I_0 \\ &= 10 \cdot 2 = 20 && \text{Definition of log} \end{aligned}$$

The loudness of the sound is 20 dB.

Now Try Exercise 97 ■

■ Natural Logarithms

Of all possible bases a for logarithms, it turns out that the most convenient choice for the purposes of calculus is the number e, which we defined in Section 4.2.

NATURAL LOGARITHM

The notation ln is an abbreviation for the Latin name *logarithmus naturalis*.

The logarithm with base e is called the **natural logarithm** and is denoted by **ln**:

$$\ln x = \log_e x$$

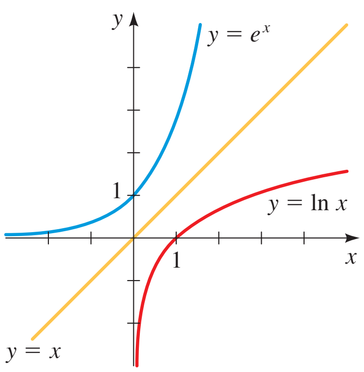


FIGURE 9 Graph of the natural logarithmic function

The natural logarithmic function $y = \ln x$ is the inverse function of the natural exponential function $y = e^x$. Both functions are graphed in Figure 9. By the definition of inverse functions we have

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

If we substitute $a = e$ and write "ln" for "$\log_e$" in the properties of logarithms mentioned earlier, we obtain the following properties of natural logarithms.

PROPERTIES OF NATURAL LOGARITHMS

Property	Reason
1. $\ln 1 = 0$	We must raise e to the power 0 to get 1.
2. $\ln e = 1$	We must raise e to the power 1 to get e.
3. $\ln e^x = x$	We must raise e to the power x to get e^x.
4. $e^{\ln x} = x$	$\ln x$ is the power to which e must be raised to get x.

Calculators are equipped with an LN key that directly gives the values of natural logarithms.

EXAMPLE 9 ■ Evaluating the Natural Logarithm Function

(a) $\ln e^8 = 8$ — Definition of natural logarithm

(b) $\ln\left(\frac{1}{e^2}\right) = \ln e^{-2} = -2$ — Definition of natural logarithm

(c) $\ln 5 \approx 1.609$ — Use LN key on calculator

Now Try Exercise 47 ■

EXAMPLE 10 ■ Finding the Domain of a Logarithmic Function

Find the domain of the function $f(x) = \ln(4 - x^2)$.

SOLUTION As with any logarithmic function, $\ln x$ is defined when $x > 0$. Thus the domain of f is

$$\{x \mid 4 - x^2 > 0\} = \{x \mid x^2 < 4\} = \{x \mid |x| < 2\}$$
$$= \{x \mid -2 < x < 2\} = (-2, 2)$$

Now Try Exercise 73 ■

EXAMPLE 11 ■ Drawing the Graph of a Logarithmic Function

Draw the graph of the function $y = x\ln(4 - x^2)$, and use it to find the asymptotes and local maximum and minimum values.

SOLUTION As in Example 10 the domain of this function is the interval $(-2, 2)$, so we choose the viewing rectangle $[-3, 3]$ by $[-3, 3]$. The graph is shown in Figure 10, and from it we see that the lines $x = -2$ and $x = 2$ are vertical asymptotes.

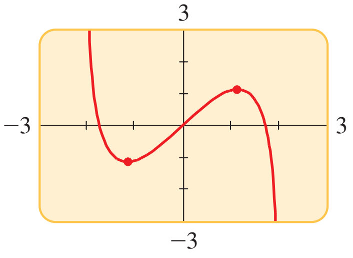


FIGURE 10
$y = x\ln(4 - x^2)$

DISCOVERY PROJECT

Orders of Magnitude

In this project we explore how to compare the sizes of real-world objects using logarithms. For example, how much bigger is an elephant than a flea? How much smaller is a man than a giant redwood? It is difficult to compare objects of such enormously varying sizes. In this project we learn how logarithms can be used to define the concept of "order of magnitude," which provides a simple and meaningful way of comparison. You can find the project at **www.stewartmath.com**.

The function has a local maximum point to the right of $x = 1$ and a local minimum point to the left of $x = -1$. By zooming in and tracing along the graph with the cursor, we find that the local maximum value is approximately 1.13 and this occurs when $x \approx 1.15$. Similarly (or by noticing that the function is odd), we find that the local minimum value is about -1.13, and it occurs when $x \approx -1.15$.

Now Try Exercise 79 ■

4.3 EXERCISES

CONCEPTS

1. $\log x$ is the exponent to which the base 10 must be raised to get ________. So we can complete the following table for $\log x$.

x	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	$10^{1/2}$
$\log x$								

2. The function $f(x) = \log_9 x$ is the logarithm function with base ________. So $f(9) =$ ________, $f(1) =$ ________, $f(\frac{1}{9}) =$ ________, $f(81) =$ ________, and $f(3) =$ ________.

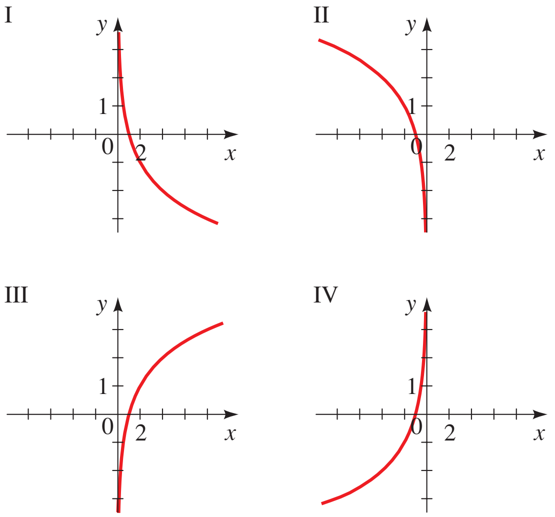
3. (a) $5^3 = 125$, so $\log_{\square} \square = \square$

(b) $\log_5 25 = 2$, so $\square^{\square} = \square$

4. Match the logarithmic function with its graph.

(a) $f(x) = \log_2 x$ **(b)** $f(x) = \log_2(-x)$

(c) $f(x) = -\log_2 x$ **(d)** $f(x) = -\log_2(-x)$



5. The natural logarithmic function $f(x) = \ln x$ has the ____________ asymptote $x =$ ____.

6. The logarithmic function $f(x) = \ln(x - 1)$ has the ____________ asymptote $x =$ ____.

SKILLS

7–8 ■ Logarithmic and Exponential Forms Complete the table by finding the appropriate logarithmic or exponential form of the equation, as in Example 1.

7.

Logarithmic form	Exponential form
$\log_8 8 = 1$	
$\log_8 64 = 2$	
	$8^{2/3} = 4$
	$8^3 = 512$
$\log_8(\frac{1}{8}) = -1$	
	$8^{-2} = \frac{1}{64}$

8.

Logarithmic form	Exponential form
	$4^3 = 64$
$\log_4 2 = \frac{1}{2}$	
	$4^{3/2} = 8$
$\log_4(\frac{1}{16}) = -2$	
$\log_4(\frac{1}{2}) = -\frac{1}{2}$	
	$4^{-5/2} = \frac{1}{32}$

9–16 ■ Exponential Form Express the equation in exponential form.

9. (a) $\log_3 81 = 4$ **(b)** $\log_3 1 = 0$

10. (a) $\log_5(\frac{1}{5}) = -1$ **(b)** $\log_4 64 = 3$

11. (a) $\log_8 2 = \frac{1}{3}$ **(b)** $\log_{10} 0.01 = -2$

12. (a) $\log_5(\frac{1}{125}) = -3$ **(b)** $\log_8 4 = \frac{2}{3}$

13. (a) $\log_3 5 = x$ **(b)** $\log_7(3y) = 2$

14. (a) $\log_6 z = 1$ **(b)** $\log_{10} 3 = 2t$

15. (a) $\ln 5 = 3y$ **(b)** $\ln(t + 1) = -1$

16. (a) $\ln(x + 1) = 2$ **(b)** $\ln(x - 1) = 4$

17–24 ■ Logarithmic Form Express the equation in logarithmic form.

17. (a) $10^4 = 10{,}000$ **(b)** $5^{-2} = \frac{1}{25}$

18. (a) $6^2 = 36$ **(b)** $10^{-1} = \frac{1}{10}$

19. **(a)** $8^{-1} = \frac{1}{8}$ **(b)** $2^{-3} = \frac{1}{8}$

20. **(a)** $4^{-3/2} = 0.125$ **(b)** $7^3 = 343$

21. **(a)** $4^x = 70$ **(b)** $3^5 = w$

22. **(a)** $3^{2x} = 10$ **(b)** $10^{-4x} = 0.1$

23. **(a)** $e^x = 2$ **(b)** $e^3 = y$

24. **(a)** $e^{x+1} = 0.5$ **(b)** $e^{0.5x} = t$

25–34 ■ Evaluating Logarithms Evaluate the expression.

25. **(a)** $\log_2 2$ **(b)** $\log_5 1$ **(c)** $\log_6 6^5$

26. **(a)** $\log_3 3^7$ **(b)** $\log_4 64$ **(c)** $\log_5 125$

27. **(a)** $\log_6 36$ **(b)** $\log_9 81$ **(c)** $\log_7 7^{10}$

28. **(a)** $\log_2 32$ **(b)** $\log_8 8^{17}$ **(c)** $\log_6 1$

29. **(a)** $\log_3\left(\frac{1}{27}\right)$ **(b)** $\log_{10} \sqrt{10}$ **(c)** $\log_5 0.2$

30. **(a)** $\log_5 125$ **(b)** $\log_{49} 7$ **(c)** $\log_9 \sqrt{3}$

31. **(a)** $3^{\log_3 5}$ **(b)** $5^{\log_5 27}$ **(c)** $e^{\ln 10}$

32. **(a)** $e^{\ln \sqrt{3}}$ **(b)** $e^{\ln(1/\pi)}$ **(c)** $10^{\log 13}$

33. **(a)** $\log_8 0.25$ **(b)** $\ln e^4$ **(c)** $\ln(1/e)$

34. **(a)** $\log_4 \sqrt{2}$ **(b)** $\log_4\left(\frac{1}{2}\right)$ **(c)** $\log_4 8$

35–44 ■ Logarithmic Equations Use the definition of the logarithmic function to find x.

35. **(a)** $\log_4 x = 3$ **(b)** $\log_{10} 0.01 = x$

36. **(a)** $\log_3 x = -2$ **(b)** $\log_5 125 = x$

37. **(a)** $\ln x = 3$ **(b)** $\ln e^2 = x$

38. **(a)** $\ln x = -1$ **(b)** $\ln(1/e) = x$

39. **(a)** $\log_7\left(\frac{1}{49}\right) = x$ **(b)** $\log_2 x = 5$

40. **(a)** $\log_4 2 = x$ **(b)** $\log_4 x = 2$

41. **(a)** $\log_2\left(\frac{1}{2}\right) = x$ **(b)** $\log_{10} x = -3$

42. **(a)** $\log_x 1000 = 3$ **(b)** $\log_x 25 = 2$

43. **(a)** $\log_x 16 = 4$ **(b)** $\log_x 8 = \frac{3}{2}$

44. **(a)** $\log_x 6 = \frac{1}{2}$ **(b)** $\log_x 3 = \frac{1}{3}$

45–48 ■ Evaluating Logarithms Use a calculator to evaluate the expression, correct to four decimal places.

45. **(a)** $\log 2$ **(b)** $\log 35.2$ **(c)** $\log\left(\frac{2}{3}\right)$

46. **(a)** $\log 50$ **(b)** $\log \sqrt{2}$ **(c)** $\log(3\sqrt{2})$

47. **(a)** $\ln 5$ **(b)** $\ln 25.3$ **(c)** $\ln(1 + \sqrt{3})$

48. **(a)** $\ln 27$ **(b)** $\ln 7.39$ **(c)** $\ln 54.6$

49–52 ■ Graphing Logarithmic Functions Sketch the graph of the function by plotting points.

49. $f(x) = \log_3 x$

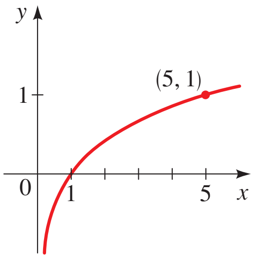
50. $g(x) = \log_4 x$

51. $f(x) = 2 \log x$

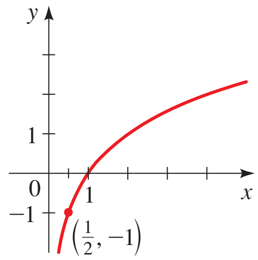
52. $g(x) = 1 + \log x$

53–56 ■ Finding Logarithmic Functions Find the function of the form $y = \log_a x$ whose graph is given.

53.



54.



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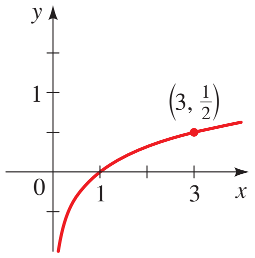

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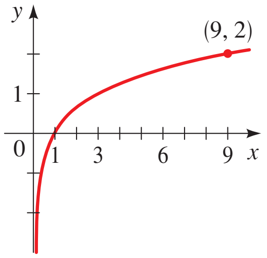
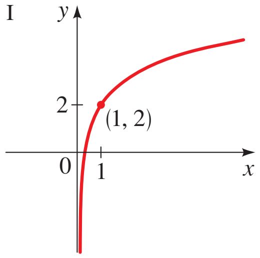


57–58 ■ Graphing Logarithmic Functions Match the logarithmic function with one of the graphs labeled I or II.

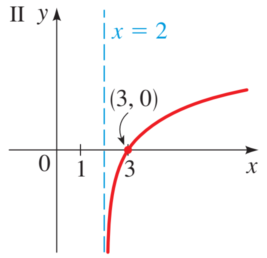
57. $f(x) = 2 + \ln x$

58. $f(x) = \ln(x - 2)$

I



II



59. **Graphing** Draw the graph of $y = 4^x$, then use it to draw the graph of $y = \log_4 x$.

60. **Graphing** Draw the graph of $y = 3^x$, then use it to draw the graph of $y = \log_3 x$.

61–72 ■ Graphing Logarithmic Functions Graph the function, not by plotting points, but by starting from the graphs in Figures 4 and 9. State the domain, range, and asymptote.

61. $g(x) = \log_5(-x)$

62. $f(x) = -\log_{10} x$

63. $f(x) = \log_2(x - 4)$

64. $g(x) = \ln(x + 2)$

65. $h(x) = \ln(x + 5)$

66. $g(x) = \log_6(x - 3)$

67. $y = 2 + \log_3 x$

68. $y = 1 - \log_{10} x$

69. $y = \log_3(x - 1) - 2$

70. $y = 1 + \ln(-x)$

71. $y = |\ln x|$

72. $y = \ln|x|$

73–78 ■ Domain Find the domain of the function.

73. $f(x) = \log_{10}(x + 3)$

74. $f(x) = \log_5(8 - 2x)$

75. $g(x) = \log_3(x^2 - 1)$

76. $g(x) = \ln(x - x^2)$

77. $h(x) = \ln x + \ln(2 - x)$

78. $h(x) = \sqrt{x - 2} - \log_5(10 - x)$

79–84 ■ Graphing Logarithmic Functions Draw the graph of the function in a suitable viewing rectangle, and use it to find the domain, the asymptotes, and the local maximum and minimum values.

79. $y = \log_{10}(1 - x^2)$

80. $y = \ln(x^2 - x)$

81. $y = x + \ln x$

82. $y = x(\ln x)^2$

83. $y = \dfrac{\ln x}{x}$

84. $y = x \log_{10}(x + 10)$

SKILLS Plus

85–88 ■ Domain of a Composition Find the functions $f \circ g$ and $g \circ f$ and their domains.

85. $f(x) = 2^x, \quad g(x) = x + 1$

86. $f(x) = 3^x, \quad g(x) = x^2 + 1$

87. $f(x) = \log_2 x, \quad g(x) = x - 2$

88. $f(x) = \log x, \quad g(x) = x^2$

89. Rates of Growth Compare the rates of growth of the functions $f(x) = \ln x$ and $g(x) = \sqrt{x}$ by drawing their graphs on a common screen using the viewing rectangle $[-1, 30]$ by $[-1, 6]$.

90. Rates of Growth

(a) By drawing the graphs of the functions

$$f(x) = 1 + \ln(1 + x) \qquad \text{and} \qquad g(x) = \sqrt{x}$$

in a suitable viewing rectangle, show that even when a logarithmic function starts out higher than a root function, it is ultimately overtaken by the root function.

(b) Find, rounded to two decimal places, the solutions of the equation $\sqrt{x} = 1 + \ln(1 + x)$.

91–92 ■ Family of Functions A family of functions is given. **(a)** Draw graphs of the family for $c = 1, 2, 3,$ and 4. **(b)** How are the graphs in part (a) related?

91. $f(x) = \log(cx)$

92. $f(x) = c \log x$

93–94 ■ Inverse Functions A function $f(x)$ is given. **(a)** Find the domain of the function f. **(b)** Find the inverse function of f.

93. $f(x) = \log_2(\log_{10} x)$

94. $f(x) = \ln(\ln(\ln x))$

95. Inverse Functions

(a) Find the inverse of the function $f(x) = \dfrac{2^x}{1 + 2^x}$.

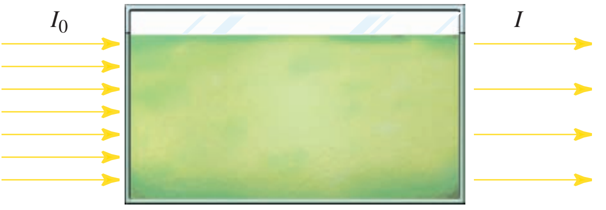
(b) What is the domain of the inverse function?

APPLICATIONS

96. Absorption of Light A spectrophotometer measures the concentration of a sample dissolved in water by shining a light through it and recording the amount of light that emerges. In other words, if we know the amount of light that is absorbed, we can calculate the concentration of the sample. For a certain substance the concentration (in moles per liter, mol/L) is found by using the formula

$$C = -2500 \ln\left(\frac{I}{I_0}\right)$$

where I_0 is the intensity of the incident light and I is the intensity of light that emerges. Find the concentration of the substance if the intensity I is 70% of I_0.



97. Carbon Dating The age of an ancient artifact can be determined by the amount of radioactive carbon-14 remaining in it. If D_0 is the original amount of carbon-14 and D is the amount remaining, then the artifact's age A (in years) is given by

$$A = -8267 \ln\left(\frac{D}{D_0}\right)$$

Find the age of an object if the amount D of carbon-14 that remains in the object is 73% of the original amount D_0.

98. Bacteria Colony A certain strain of bacteria divides every 3 hours. If a colony is started with 50 bacteria, then the time t (in hours) required for the colony to grow to N bacteria is given by

$$t = 3\frac{\log(N/50)}{\log 2}$$

Find the time required for the colony to grow to a million bacteria.

99. Investment The time required to double the amount of an investment at an interest rate r compounded continuously is given by

$$t = \frac{\ln 2}{r}$$

Find the time required to double an investment at 6%, 7%, and 8%.

100. Charging a Battery The rate at which a battery charges is slower the closer the battery is to its maximum charge C_0. The time (in hours) required to charge a fully discharged battery to a charge C is given by

$$t = -k \ln\left(1 - \frac{C}{C_0}\right)$$

where k is a positive constant that depends on the battery. For a certain battery, $k = 0.25$. If this battery is fully discharged, how long will it take to charge to 90% of its maximum charge C_0?