101. Difficulty of a Task The difficulty in "acquiring a target" (such as using your mouse to click on an icon on your computer screen) depends on the distance to the target and the size of the target. According to Fitts's Law, the index of difficulty (ID) is given by

$$\text{ID} = \frac{\log(2A/W)}{\log 2}$$

where W is the width of the target and A is the distance to the center of the target. Compare the difficulty of clicking on an icon that is 5 mm wide to clicking on one that is 10 mm wide. In each case, assume that the mouse is 100 mm from the icon.

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

102. DISCUSS: The Height of the Graph of a Logarithmic Function Suppose that the graph of $y = 2^x$ is drawn on a coordinate plane where the unit of measurement is an inch.

(a) Show that at a distance 2 ft to the right of the origin the height of the graph is about 265 mi.

(b) If the graph of $y = \log_2 x$ is drawn on the same set of axes, how far to the right of the origin do we have to go before the height of the curve reaches 2 ft?

103. DISCUSS: The Googolplex A **googol** is 10^{100}, and a **googolplex** is 10^{googol}. Find

$$\log(\log(\text{googol})) \qquad \text{and} \qquad \log(\log(\log(\text{googolplex})))$$

104. DISCUSS: Comparing Logarithms Which is larger, $\log_4 17$ or $\log_5 24$? Explain your reasoning.

105. DISCUSS ■ DISCOVER: The Number of Digits in an Integer Compare log 1000 to the number of digits in 1000. Do the same for 10,000. How many digits does any number between 1000 and 10,000 have? Between what two values must the common logarithm of such a number lie? Use your observations to explain why the number of digits in any positive integer x is $[\![\log x]\!] + 1$. (The symbol $[\![n]\!]$ is the greatest integer function defined in Section 2.2.) How many digits does the number 2^{100} have?

4.4 LAWS OF LOGARITHMS

■ Laws of Logarithms ■ Expanding and Combining Logarithmic Expressions
■ Change of Base Formula

In this section we study properties of logarithms. These properties give logarithmic functions a wide range of applications, as we will see in Sections 4.6 and 4.7.

■ Laws of Logarithms

Since logarithms are exponents, the Laws of Exponents give rise to the Laws of Logarithms.

LAWS OF LOGARITHMS

Let a be a positive number, with $a \neq 1$. Let A, B, and C be any real numbers with $A > 0$ and $B > 0$.

Law	Description
1. $\log_a(AB) = \log_a A + \log_a B$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
2. $\log_a\left(\dfrac{A}{B}\right) = \log_a A - \log_a B$	The logarithm of a quotient of numbers is the difference of the logarithms of the numbers.
3. $\log_a(A^C) = C \log_a A$	The logarithm of a power of a number is the exponent times the logarithm of the number.

Proof We make use of the property $\log_a a^x = x$ from Section 4.3.

Law 1 Let $\log_a A = u$ and $\log_a B = v$. When written in exponential form, these equations become

$$a^u = A \qquad \text{and} \qquad a^v = B$$

Thus
$$\log_a(AB) = \log_a(a^u a^v) = \log_a(a^{u+v})$$
$$= u + v = \log_a A + \log_a B$$

Law 2 Using Law 1, we have

$$\log_a A = \log_a\left[\left(\frac{A}{B}\right)B\right] = \log_a\left(\frac{A}{B}\right) + \log_a B$$

so
$$\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$$

Law 3 Let $\log_a A = u$. Then $a^u = A$, so

$$\log_a(A^C) = \log_a(a^u)^C = \log_a(a^{uC}) = uC = C\log_a A$$ ■

EXAMPLE 1 ■ Using the Laws of Logarithms to Evaluate Expressions

Evaluate each expression.

(a) $\log_4 2 + \log_4 32$

(b) $\log_2 80 - \log_2 5$

(c) $-\frac{1}{3}\log 8$

SOLUTION

(a) $\log_4 2 + \log_4 32 = \log_4(2 \cdot 32)$ — Law 1

$= \log_4 64 = 3$ — Because $64 = 4^3$

(b) $\log_2 80 - \log_2 5 = \log_2\left(\frac{80}{5}\right)$ — Law 2

$= \log_2 16 = 4$ — Because $16 = 2^4$

(c) $-\frac{1}{3}\log 8 = \log 8^{-1/3}$ — Law 3

$= \log\left(\frac{1}{2}\right)$ — Property of negative exponents

≈ -0.301 — Calculator

Now Try Exercises 9, 11, and 13 ■

■ Expanding and Combining Logarithmic Expressions

The Laws of Logarithms allow us to write the logarithm of a product or a quotient as the sum or difference of logarithms. This process, called *expanding* a logarithmic expression, is illustrated in the next example.

EXAMPLE 2 ■ Expanding Logarithmic Expressions

Use the Laws of Logarithms to expand each expression.

(a) $\log_2(6x)$ **(b)** $\log_5(x^3y^6)$ **(c)** $\ln\left(\frac{ab}{\sqrt[3]{c}}\right)$

SOLUTION

(a) $\log_2(6x) = \log_2 6 + \log_2 x$ — Law 1

(b) $\log_5(x^3y^6) = \log_5 x^3 + \log_5 y^6$ — Law 1

$= 3\log_5 x + 6\log_5 y$ — Law 3

(c) $\ln\left(\dfrac{ab}{\sqrt[3]{c}}\right) = \ln(ab) - \ln\sqrt[3]{c}$ Law 2

$= \ln a + \ln b - \ln c^{1/3}$ Law 1

$= \ln a + \ln b - \frac{1}{3}\ln c$ Law 3

Now Try Exercises 23, 31, and 37 ■

The Laws of Logarithms also allow us to reverse the process of expanding that was done in Example 2. That is, we can write sums and differences of logarithms as a single logarithm. This process, called *combining* logarithmic expressions, is illustrated in the next example.

EXAMPLE 3 ■ Combining Logarithmic Expressions

Use the Laws of Logarithms to combine each expression into a single logarithm.

(a) $3\log x + \frac{1}{2}\log(x+1)$

(b) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1)$

SOLUTION

(a) $3\log x + \frac{1}{2}\log(x+1) = \log x^3 + \log(x+1)^{1/2}$ Law 3

$= \log(x^3(x+1)^{1/2})$ Law 1

(b) $3\ln s + \frac{1}{2}\ln t - 4\ln(t^2+1) = \ln s^3 + \ln t^{1/2} - \ln(t^2+1)^4$ Law 3

$= \ln(s^3 t^{1/2}) - \ln(t^2+1)^4$ Law 1

$= \ln\left(\dfrac{s^3\sqrt{t}}{(t^2+1)^4}\right)$ Law 2

Now Try Exercises 51 and 53 ■

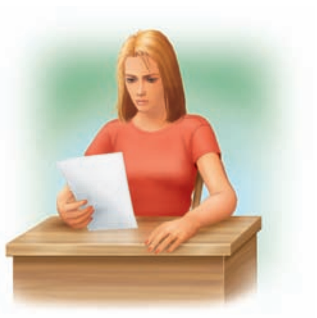
Warning Although the Laws of Logarithms tell us how to compute the logarithm of a product or a quotient, *there is no corresponding rule for the logarithm of a sum or a difference*. For instance,

$$\log_a(x+y) \neq \log_a x + \log_a y$$

In fact, we know that the right side is equal to $\log_a(xy)$. Also, don't improperly simplify quotients or powers of logarithms. For instance,

$$\frac{\log 6}{\log 2} \neq \log\left(\frac{6}{2}\right) \quad \text{and} \quad (\log_2 x)^3 \neq 3\log_2 x$$

Logarithmic functions are used to model a variety of situations involving human behavior. One such behavior is how quickly we forget things we have learned. For example, if you learn algebra at a certain performance level (say, 90% on a test) and then don't use algebra for a while, how much will you retain after a week, a month, or a year? Hermann Ebbinghaus (1850–1909) studied this phenomenon and formulated the law described in the next example.



Forgetting what we've learned depends on how long ago we learned it.

EXAMPLE 4 ■ The Law of Forgetting

If a task is learned at a performance level P_0, then after a time interval t the performance level P satisfies

$$\log P = \log P_0 - c\log(t+1)$$

where c is a constant that depends on the type of task and t is measured in months.

(a) Solve for P.

(b) If your score on a history test is 90, what score would you expect to get on a similar test after two months? After a year? (Assume that $c = 0.2$.)

SOLUTION

(a) We first combine the right-hand side.

$$\log P = \log P_0 - c\log(t+1) \qquad \text{Given equation}$$

$$\log P = \log P_0 - \log(t+1)^c \qquad \text{Law 3}$$

$$\log P = \log \frac{P_0}{(t+1)^c} \qquad \text{Law 2}$$

$$P = \frac{P_0}{(t+1)^c} \qquad \text{Because log is one-to-one}$$

(b) Here $P_0 = 90$, $c = 0.2$, and t is measured in months.

In 2 months: $t = 2$ and $P = \dfrac{90}{(2+1)^{0.2}} \approx 72$

In 1 year: $t = 12$ and $P = \dfrac{90}{(12+1)^{0.2}} \approx 54$

Your expected scores after 2 months and after 1 year are 72 and 54, respectively.

Now Try Exercise 73 ■

■ Change of Base Formula

For some purposes we find it useful to change from logarithms in one base to logarithms in another base. Suppose we are given $\log_a x$ and want to find $\log_b x$. Let

$$y = \log_b x$$

We write this in exponential form and take the logarithm, with base a, of each side.

$$b^y = x \qquad \text{Exponential form}$$

$$\log_a(b^y) = \log_a x \qquad \text{Take } \log_a \text{ of each side}$$

$$y\log_a b = \log_a x \qquad \text{Law 3}$$

$$y = \frac{\log_a x}{\log_a b} \qquad \text{Divide by } \log_a b$$

This proves the following formula.

We may write the Change of Base Formula as

$$\log_b x = \left(\frac{1}{\log_a b}\right)\log_a x$$

So $\log_b x$ is just a constant multiple of $\log_a x$; the constant is $\dfrac{1}{\log_a b}$.

CHANGE OF BASE FORMULA

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular, if we put $x = a$, then $\log_a a = 1$, and this formula becomes

$$\log_b a = \frac{1}{\log_a b}$$

We can now evaluate a logarithm to *any* base by using the Change of Base Formula to express the logarithm in terms of common logarithms or natural logarithms and then using a calculator.

EXAMPLE 5 ■ Evaluating Logarithms with the Change of Base Formula

Use the Change of Base Formula and common or natural logarithms to evaluate each logarithm, rounded to five decimal places.

(a) $\log_8 5$ **(b)** $\log_9 20$

SOLUTION

(a) We use the Change of Base Formula with $b = 8$ and $a = 10$:

$$\log_8 5 = \frac{\log_{10} 5}{\log_{10} 8} \approx 0.77398$$

(b) We use the Change of Base Formula with $b = 9$ and $a = e$:

$$\log_9 20 = \frac{\ln 20}{\ln 9} \approx 1.36342$$

Now Try Exercises 59 and 61 ■

EXAMPLE 6 ■ Using the Change of Base Formula to Graph a Logarithmic Function

Use a graphing calculator to graph $f(x) = \log_6 x$.

SOLUTION Calculators don't have a key for $\log_6$, so we use the Change of Base Formula to write

$$f(x) = \log_6 x = \frac{\ln x}{\ln 6}$$

Since calculators do have an LN key, we can enter this new form of the function and graph it. The graph is shown in Figure 1.

Now Try Exercise 67 ■

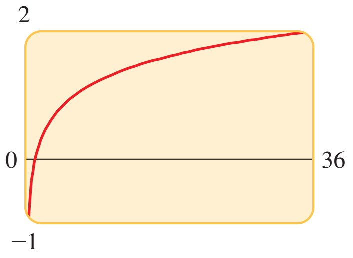


FIGURE 1

$f(x) = \log_6 x = \dfrac{\ln x}{\ln 6}$

4.4 EXERCISES

CONCEPTS

1. The logarithm of a product of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5(25 \cdot 125) =$ ________ + ________.

2. The logarithm of a quotient of two numbers is the same as the ________ of the logarithms of these numbers. So $\log_5(\frac{25}{125}) =$ ________ − ________.

3. The logarithm of a number raised to a power is the same as the ________ times the logarithm of the number. So $\log_5(25^{10}) =$ ________ · ________.

4. We can expand $\log\left(\dfrac{x^2 y}{z}\right)$ to get ________________.

5. We can combine $2\log x + \log y - \log z$ to get ________.

6. (a) Most calculators can find logarithms with base ________ and base ________. To find logarithms with different bases, we use the ________________ Formula. To find $\log_7 12$, we write

$$\log_7 12 = \frac{\log \square}{\log \square} \approx \_\_\_\_\_\_\_\_$$

(b) Do we get the same answer if we perform the calculation in part (a) using ln in place of log?

7–8 ■ *True or False*?

7. (a) $\log(A + B)$ is the same as $\log A + \log B$.

(b) $\log AB$ is the same as $\log A + \log B$.

8. (a) $\log \dfrac{A}{B}$ is the same as $\log A - \log B$.

(b) $\dfrac{\log A}{\log B}$ is the same as $\log A - \log B$.

SKILLS

9–22 ■ Evaluating Logarithms Use the Laws of Logarithms to evaluate the expression.

9. $\log 50 + \log 200$

10. $\log_6 9 + \log_6 24$

11. $\log_2 60 - \log_2 15$

12. $\log_3 135 - \log_3 45$

13. $\frac{1}{4}\log_3 81$

14. $-\frac{1}{3}\log_3 27$

15. $\log_5 \sqrt{5}$

16. $\log_5 \dfrac{1}{\sqrt{125}}$

17. $\log_2 6 - \log_2 15 + \log_2 20$

18. $\log_3 100 - \log_3 18 - \log_3 50$

19. $\log_4 16^{100}$

20. $\log_2 8^{33}$

21. $\log(\log 10^{10,000})$

22. $\ln(\ln e^{e^{200}})$

23–48 ■ Expanding Logarithmic Expressions Use the Laws of Logarithms to expand the expression.

23. $\log_3 8x$

24. $\log_6 7r$

25. $\log_3 2xy$

26. $\log_5 4st$

27. $\ln a^3$

28. $\log \sqrt{t^5}$

29. $\log_2(xy)^{10}$

30. $\ln \sqrt{ab}$

31. $\log_2(AB^2)$

32. $\log_3(x\sqrt{y})$

33. $\log_3 \dfrac{2x}{y}$

34. $\ln \dfrac{r}{3s}$

35. $\log_5\left(\dfrac{3x^2}{y^3}\right)$

36. $\log_2\left(\dfrac{s^5}{7t^2}\right)$

37. $\log_3 \dfrac{\sqrt{3x^5}}{y}$

38. $\log \dfrac{y^3}{\sqrt{2x}}$

39. $\log\left(\dfrac{x^3y^4}{z^6}\right)$

40. $\log_a\left(\dfrac{x^2}{yz^3}\right)$

41. $\ln \sqrt{x^4 + 2}$

42. $\log \sqrt[3]{x^2 + 4}$

43. $\ln\left(x\sqrt{\dfrac{y}{z}}\right)$

44. $\ln \dfrac{3x^2}{(x + 1)^{10}}$

45. $\log \sqrt[4]{x^2 + y^2}$

46. $\log\left(\dfrac{x}{\sqrt[3]{1 - x}}\right)$

47. $\log \sqrt{\dfrac{x^2 + 4}{(x^2 + 1)(x^3 - 7)^2}}$

48. $\log \sqrt{x\sqrt{y\sqrt{z}}}$

49–58 ■ Combining Logarithmic Expressions Use the Laws of Logarithms to combine the expression.

49. $\log_4 6 + 2\log_4 7$

50. $\frac{1}{2}\log_2 5 - 2\log_2 7$

51. $2\log x - 3\log(x + 1)$

52. $3\ln 2 + 2\ln x - \frac{1}{2}\ln(x + 4)$

53. $4\log x - \frac{1}{3}\log(x^2 + 1) + 2\log(x - 1)$

54. $\log_5(x^2 - 1) - \log_5(x - 1)$

55. $\ln(a + b) + \ln(a - b) - 2\ln c$

56. $2(\log_5 x + 2\log_5 y - 3\log_5 z)$

57. $\frac{1}{3}\log(x + 2)^3 + \frac{1}{2}[\log x^4 - \log(x^2 - x - 6)^2]$

58. $\log_a b + c\log_a d - r\log_a s$

59–66 ■ Change of Base Formula Use the Change of Base Formula and a calculator to evaluate the logarithm, rounded to six decimal places. Use either natural or common logarithms.

59. $\log_2 5$

60. $\log_5 2$

61. $\log_3 16$

62. $\log_6 92$

63. $\log_7 2.61$

64. $\log_6 532$

65. $\log_4 125$

66. $\log_{12} 2.5$

67. Change of Base Formula Use the Change of Base Formula to show that

$$\log_3 x = \frac{\ln x}{\ln 3}$$

Then use this fact to draw the graph of the function $f(x) = \log_3 x$.

SKILLS Plus

68. Families of Functions Draw graphs of the family of functions $y = \log_a x$ for $a = 2, e, 5$, and 10 on the same screen, using the viewing rectangle $[0, 5]$ by $[-3, 3]$. How are these graphs related?

69. Change of Base Formula Use the Change of Base Formula to show that

$$\log e = \frac{1}{\ln 10}$$

70. Change of Base Formula Simplify: $(\log_2 5)(\log_5 7)$

71. A Logarithmic Identity Show that

$$-\ln(x - \sqrt{x^2 - 1}) = \ln(x + \sqrt{x^2 - 1})$$

APPLICATIONS

72. Forgetting Use the Law of Forgetting (Example 4) to estimate a student's score on a biology test two years after he got a score of 80 on a test covering the same material. Assume that $c = 0.3$ and t is measured in months.

73. Wealth Distribution Vilfredo Pareto (1848–1923) observed that most of the wealth of a country is owned by a few members of the population. **Pareto's Principle** is

$$\log P = \log c - k\log W$$

where W is the wealth level (how much money a person has) and P is the number of people in the population having that much money.

(a) Solve the equation for P.

(b) Assume that $k = 2.1$ and $c = 8000$, and that W is measured in millions of dollars. Use part (a) to find the number of people who have \$2 million or more. How many people have \$10 million or more?