6.1 ANGLE MEASURE

■ Angle Measure ■ Angles in Standard Position ■ Length of a Circular Arc ■ Area of a Circular Sector ■ Circular Motion

An **angle** AOB consists of two rays R_1 and R_2 with a common vertex O (see Figure 1). We often interpret an angle as a rotation of the ray R_1 onto R_2. In this case R_1 is called the **initial side**, and R_2 is called the **terminal side** of the angle. If the rotation is counterclockwise, the angle is considered **positive**, and if the rotation is clockwise, the angle is considered **negative**.

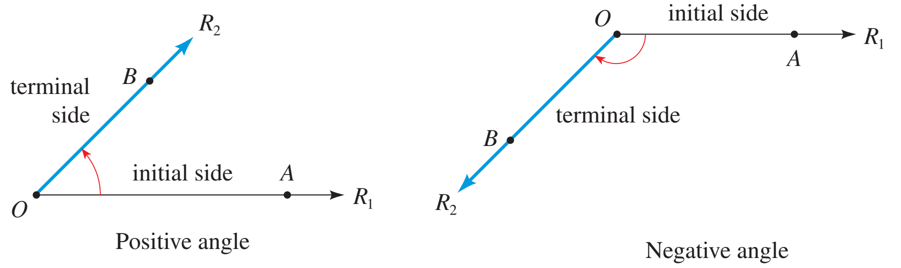


FIGURE 1

■ Angle Measure

The **measure** of an angle is the amount of rotation about the vertex required to move R_1 onto R_2. Intuitively, this is how much the angle "opens." One unit of measurement for angles is the **degree**. An angle of measure 1 degree is formed by rotating the initial side $\frac{1}{360}$ of a complete revolution. In calculus and other branches of mathematics a more natural method of measuring angles is used: *radian measure*. The amount an angle opens is measured along the arc of a circle of radius 1 with its center at the vertex of the angle.

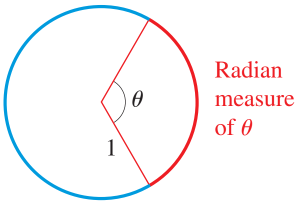


FIGURE 2

DEFINITION OF RADIAN MEASURE

If a circle of radius 1 is drawn with the vertex of an angle at its center, then the measure of this angle in **radians** (abbreviated **rad**) is the length of the arc that subtends the angle (see Figure 2).

The circumference of the circle of radius 1 is 2π, so a complete revolution has measure 2π rad, a straight angle has measure π rad, and a right angle has measure $\pi/2$ rad. An angle that is subtended by an arc of length 2 along the unit circle has radian measure 2 (see Figure 3).

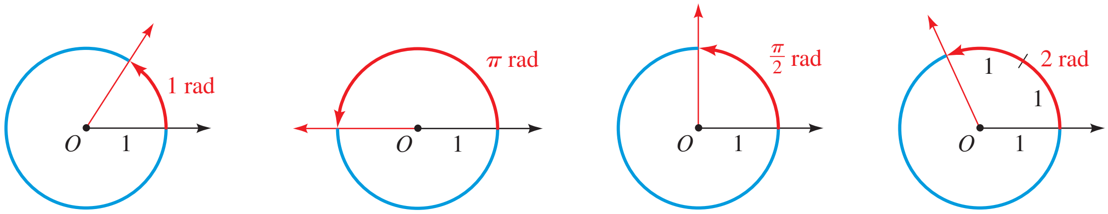


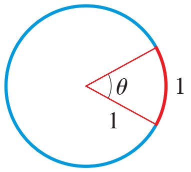
FIGURE 3 Radian measure

Since a complete revolution measured in degrees is 360° and measured in radians is 2π rad, we get the following simple relationship between these two methods of angle measurement.

RELATIONSHIP BETWEEN DEGREES AND RADIANS

$$180° = \pi \text{ rad} \qquad 1 \text{ rad} = \left(\frac{180}{\pi}\right)^{\circ} \qquad 1° = \frac{\pi}{180} \text{ rad}$$

1. To convert degrees to radians, multiply by $\frac{\pi}{180}$.
2. To convert radians to degrees, multiply by $\frac{180}{\pi}$.



Measure of $\theta = 1$ rad
Measure of $\theta \approx 57.296°$

FIGURE 4

To get some idea of the size of a radian, notice that

$$1 \text{ rad} \approx 57.296° \qquad \text{and} \qquad 1° \approx 0.01745 \text{ rad}$$

An angle θ of measure 1 rad is shown in Figure 4.

EXAMPLE 1 ■ Converting Between Radians and Degrees

(a) Express 60° in radians. **(b)** Express $\frac{\pi}{6}$ rad in degrees.

SOLUTION The relationship between degrees and radians gives

(a) $60° = 60\left(\frac{\pi}{180}\right) \text{ rad} = \frac{\pi}{3} \text{ rad}$ **(b)** $\frac{\pi}{6} \text{ rad} = \left(\frac{\pi}{6}\right)\left(\frac{180}{\pi}\right) = 30°$

Now Try Exercises 5 and 17 ■

A note on terminology: We often use a phrase such as "a 30° angle" to mean *an angle whose measure is* 30°. Also, for an angle θ we write $\theta = 30°$ or $\theta = \pi/6$ to mean *the measure of* θ *is* 30° *or* $\pi/6$ *rad*. When no unit is given, the angle is assumed to be measured in radians.

■ Angles in Standard Position

An angle is in **standard position** if it is drawn in the xy-plane with its vertex at the origin and its initial side on the positive x-axis. Figure 5 gives examples of angles in standard position.

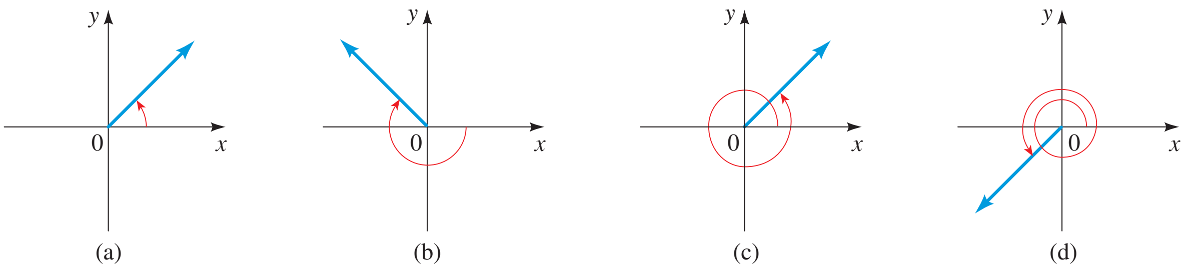


FIGURE 5 Angles in standard position

Two angles in standard position are **coterminal** if their sides coincide. In Figure 5 the angles in (a) and (c) are coterminal.

EXAMPLE 2 ■ Coterminal Angles

(a) Find angles that are coterminal with the angle $\theta = 30°$ in standard position.

(b) Find angles that are coterminal with the angle $\theta = \frac{\pi}{3}$ in standard position.

SOLUTION

(a) To find positive angles that are coterminal with θ, we add any multiple of $360°$. Thus

$$30° + 360° = 390° \qquad \text{and} \qquad 30° + 720° = 750°$$

are coterminal with $\theta = 30°$. To find negative angles that are coterminal with θ, we subtract any multiple of $360°$. Thus

$$30° - 360° = -330° \qquad \text{and} \qquad 30° - 720° = -690°$$

are coterminal with θ. (See Figure 6.)

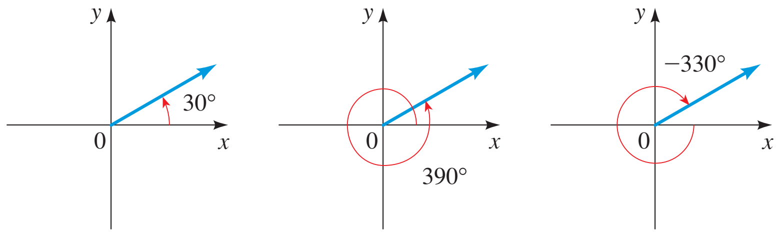


FIGURE 6

(b) To find positive angles that are coterminal with θ, we add any multiple of 2π. Thus

$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3} \qquad \text{and} \qquad \frac{\pi}{3} + 4\pi = \frac{13\pi}{3}$$

are coterminal with $\theta = \pi/3$. To find negative angles that are coterminal with θ, we subtract any multiple of 2π. Thus

$$\frac{\pi}{3} - 2\pi = -\frac{5\pi}{3} \qquad \text{and} \qquad \frac{\pi}{3} - 4\pi = -\frac{11\pi}{3}$$

are coterminal with θ. (See Figure 7.)

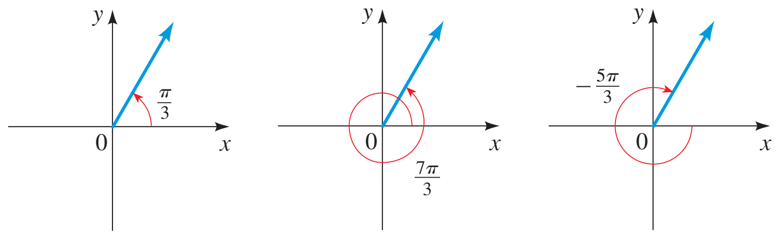


FIGURE 7

Now Try Exercises 29 and 31 ■

EXAMPLE 3 ■ Coterminal Angles

Find an angle with measure between $0°$ and $360°$ that is coterminal with the angle of measure $1290°$ in standard position.

SOLUTION We can subtract $360°$ as many times as we wish from $1290°$, and the resulting angle will be coterminal with $1290°$. Thus $1290° - 360° = 930°$ is coterminal with $1290°$, and so is the angle $1290° - 2(360)° = 570°$.

To find the angle we want between $0°$ and $360°$, we subtract $360°$ from $1290°$ as many times as necessary. An efficient way to do this is to determine how many times $360°$ goes into $1290°$, that is, divide 1290 by 360, and the remainder will be the angle we are looking for. We see that 360 goes into 1290 three times with a remainder of 210. Thus 210° is the desired angle (see Figure 8).

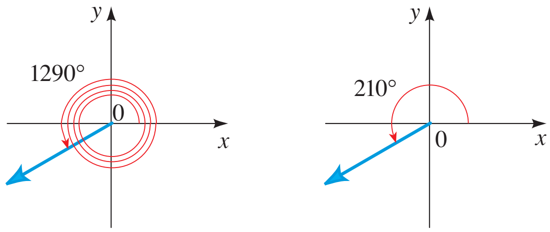


FIGURE 8

Now Try Exercise 41 ■

Length of a Circular Arc

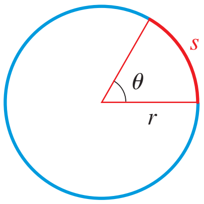


FIGURE 9 $s = \theta r$

An angle whose radian measure is θ is subtended by an arc that is the fraction $\theta/(2\pi)$ of the circumference of a circle. Thus in a circle of radius r the length s of an arc that subtends the angle θ (see Figure 9) is

$$s = \frac{\theta}{2\pi} \times \text{circumference of circle}$$

$$= \frac{\theta}{2\pi}(2\pi r) = \theta r$$

LENGTH OF A CIRCULAR ARC

In a circle of radius r the length s of an arc that subtends a central angle of θ radians is

$$s = r\theta$$

Solving for θ, we get the important formula

$$\theta = \frac{s}{r}$$

This formula allows us to define radian measure using a circle of any radius r: The radian measure of an angle θ is s/r, where s is the length of the circular arc that subtends θ in a circle of radius r (see Figure 10).

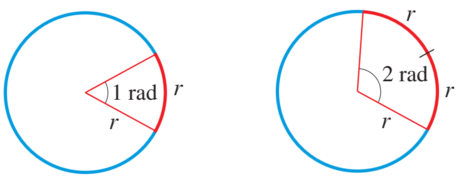


FIGURE 10 The radian measure of θ is the number of "radiuses" that can fit in the arc that subtends θ; hence the term *radian*.

EXAMPLE 4 ■ Arc Length and Angle Measure

(a) Find the length of an arc of a circle with radius 10 m that subtends a central angle of 30°.

(b) A central angle θ in a circle of radius 4 m is subtended by an arc of length 6 m. Find the measure of θ in radians.

SOLUTION

(a) From Example 1(b) we see that $30° = \pi/6$ rad. So the length of the arc is

The formula $s = r\theta$ is true only when θ is measured in radians.

$$s = r\theta = (10)\frac{\pi}{6} = \frac{5\pi}{3} \text{ m}$$

(b) By the formula $\theta = s/r$ we have

$$\theta = \frac{s}{r} = \frac{6}{4} = \frac{3}{2} \text{ rad}$$

Now Try Exercises 57 and 59 ■

Area of a Circular Sector

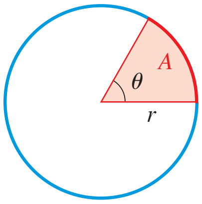


FIGURE 11
$A = \frac{1}{2}r^2\theta$

The area of a circle of radius r is $A = \pi r^2$. A sector of this circle with central angle θ has an area that is the fraction $\theta/(2\pi)$ of the area of the entire circle (see Figure 11). So the area of this sector is

$$A = \frac{\theta}{2\pi} \times \text{area of circle}$$

$$= \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta$$

AREA OF A CIRCULAR SECTOR

In a circle of radius r the area A of a sector with a central angle of θ radians is

$$A = \frac{1}{2}r^2\theta$$

EXAMPLE 5 ■ Area of a Sector

Find the area of a sector of a circle with central angle 60° if the radius of the circle is 3 m.

SOLUTION To use the formula for the area of a circular sector, we must find the central angle of the sector in radians: $60° = 60(\pi/180)$ rad $= \pi/3$ rad. Thus the area of the sector is

The formula $A = \frac{1}{2}r^2\theta$ is true only when θ is measured in radians.

$$A = \frac{1}{2}r^2\theta = \frac{1}{2}(3)^2\left(\frac{\pi}{3}\right) = \frac{3\pi}{2} \text{ m}^2$$

Now Try Exercise 63 ■

Circular Motion

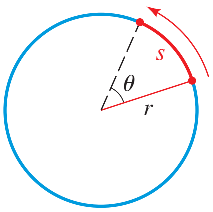


FIGURE 12

Suppose a point moves along a circle as shown in Figure 12. There are two ways to describe the motion of the point: linear speed and angular speed. **Linear speed** is the rate at which the distance traveled is changing, so linear speed is the distance traveled divided by the time elapsed. **Angular speed** is the rate at which the central angle θ is changing, so angular speed is the number of radians this angle changes divided by the time elapsed.

LINEAR SPEED AND ANGULAR SPEED

Suppose a point moves along a circle of radius r and the ray from the center of the circle to the point traverses θ radians in time t. Let $s = r\theta$ be the distance the point travels in time t. Then the speed of the object is given by

$$\textbf{Angular speed} \qquad \omega = \frac{\theta}{t}$$

$$\textbf{Linear speed} \qquad v = \frac{s}{t}$$

The symbol ω is the Greek letter "omega."

EXAMPLE 6 ■ Finding Linear and Angular Speed

A boy rotates a stone in a 3-ft-long sling at the rate of 15 revolutions every 10 seconds. Find the angular and linear velocities of the stone.

SOLUTION In 10 s the angle θ changes by $15 \cdot 2\pi = 30\pi$ rad. So the *angular speed* of the stone is

$$\omega = \frac{\theta}{t} = \frac{30\pi \text{ rad}}{10 \text{ s}} = 3\pi \text{ rad/s}$$

The distance traveled by the stone in 10 s is $s = 15 \cdot 2\pi r = 15 \cdot 2\pi \cdot 3 = 90\pi$ ft. So the *linear speed* of the stone is

$$v = \frac{s}{t} = \frac{90\pi \text{ ft}}{10 \text{ s}} = 9\pi \text{ ft/s}$$

Now Try Exercise 85 ■

Notice that angular speed does *not* depend on the radius of the circle; it depends only on the angle θ. However, if we know the angular speed ω and the radius r, we can find linear speed as follows: $v = s/t = r\theta/t = r(\theta/t) = r\omega$.

RELATIONSHIP BETWEEN LINEAR AND ANGULAR SPEED

If a point moves along a circle of radius r with angular speed ω, then its linear speed v is given by

$$v = r\omega$$

EXAMPLE 7 ■ Finding Linear Speed from Angular Speed

A woman is riding a bicycle whose wheels are 26 in. in diameter. If the wheels rotate at 125 revolutions per minute (rpm), find the speed (in mi/h) at which she is traveling.

SOLUTION The angular speed of the wheels is $2\pi \cdot 125 = 250\pi$ rad/min. Since the wheels have radius 13 in. (half the diameter), the linear speed is

$$v = r\omega = 13 \cdot 250\pi \approx 10{,}210.2 \text{ in./min}$$

Since there are 12 inches per foot, 5280 feet per mile, and 60 minutes per hour, her speed in miles per hour is

$$\frac{10{,}210.2 \text{ in./min} \times 60 \text{ min/h}}{12 \text{ in./ft} \times 5280 \text{ ft/mi}} = \frac{612{,}612 \text{ in./h}}{63{,}360 \text{ in./mi}}$$
$$\approx 9.7 \text{ mi/h}$$

Now Try Exercise 87 ■

6.1 EXERCISES

CONCEPTS

1. **(a)** The radian measure of an angle θ is the length of the ________ that subtends the angle in a circle of radius ________.

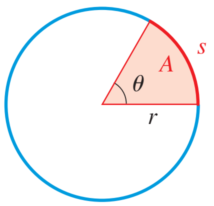
(b) To convert degrees to radians, we multiply by ________.

(c) To convert radians to degrees, we multiply by ________.

2. A central angle θ is drawn in a circle of radius r, as in the figure below.

(a) The length of the arc subtended by θ is $s =$ ________.

(b) The area of the sector with central angle θ is $A =$ ________.

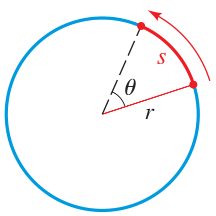


3. Suppose a point moves along a circle with radius r as shown in the figure below. The point travels a distance s along the circle in time t.

(a) The angular speed of the point is $\omega = \frac{\square}{\square}$.

(b) The linear speed of the point is $v = \frac{\square}{\square}$.

(c) The linear speed v and the angular speed ω are related by the equation $v =$ ________.



4. Object A is traveling along a circle of radius 2, and Object B is traveling along a circle of radius 5. The objects have the same angular speed. Do the objects have the same linear speed? If not, which object has the greater linear speed?

SKILLS

5–16 ■ From Degrees to Radians Find the radian measure of the angle with the given degree measure. Round your answer to three decimal places.

5. 15° **6.** 36° **7.** 54° **8.** 75°

9. −45° **10.** −30° **11.** 100° **12.** 200°

13. 1000° **14.** 3600° **15.** −70° **16.** −150°

17–28 ■ From Radians to Degrees Find the degree measure of the angle with the given radian measure.

17. $\frac{5\pi}{3}$ **18.** $\frac{3\pi}{4}$ **19.** $\frac{5\pi}{6}$

20. $-\frac{3\pi}{2}$ **21.** 3 **22.** −2

23. −1.2 **24.** 3.4 **25.** $\frac{\pi}{10}$

26. $\frac{5\pi}{18}$ **27.** $-\frac{2\pi}{15}$ **28.** $-\frac{13\pi}{12}$

29–34 ■ Coterminal Angles The measure of an angle in standard position is given. Find two positive angles and two negative angles that are coterminal with the given angle.

29. 50° **30.** 135° **31.** $\frac{3\pi}{4}$

32. $\frac{11\pi}{6}$ **33.** $-\frac{\pi}{4}$ **34.** −45°

35–40 ■ Coterminal Angles? The measures of two angles in standard position are given. Determine whether the angles are coterminal.

35. 70°, 430° **36.** −30°, 330°

37. $\frac{5\pi}{6}, \frac{17\pi}{6}$ **38.** $\frac{32\pi}{3}, \frac{11\pi}{3}$

39. 155°, 875° **40.** 50°, 340°

41–46 ■ Finding a Coterminal Angle Find an angle between 0° and 360° that is coterminal with the given angle.

41. 400° **42.** 375°

43. 780° **44.** −100°

45. −800° **46.** 1270°

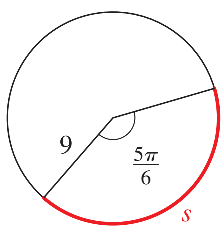
47–52 ■ Finding a Coterminal Angle Find an angle between 0 and 2π that is coterminal with the given angle.

47. $\frac{19\pi}{6}$ **48.** $-\frac{5\pi}{3}$ **49.** 25π

50. 10 **51.** $\frac{17\pi}{4}$ **52.** $\frac{51\pi}{2}$

53–62 ■ Circular Arcs Find the length s of the circular arc, the radius r of the circle, or the central angle θ, as indicated.

53.



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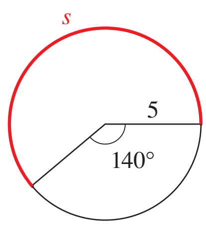

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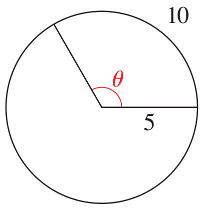

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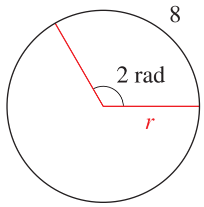


57. Find the length s of the arc that subtends a central angle of measure 3 rad in a circle of radius 5 cm.

58. Find the length s of the arc that subtends a central angle of measure 40° in a circle of radius 12 m.

59. A central angle θ in a circle of radius 9 m is subtended by an arc of length 14 m. Find the measure of θ in degrees and radians.

60. An arc of length 15 ft subtends a central angle θ in a circle of radius 9 ft. Find the measure of θ in degrees and radians.

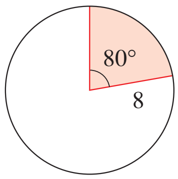
61. Find the radius r of the circle if an arc of length 15 m on the circle subtends a central angle of $5\pi/6$.

62. Find the radius r of the circle if an arc of length 20 cm on the circle subtends a central angle of 50°.

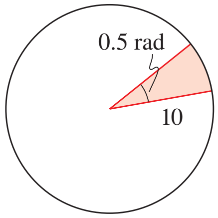
63–70 ■ Area of a Circular Sector These exercises involve the formula for the area of a circular sector.

63. Find the area of the sector shown in each figure.

(a)

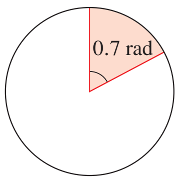


(b)



64. Find the radius of each circle if the area of the sector is 12.

(a)



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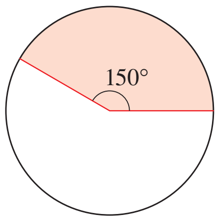


65. Find the area of a sector with central angle $2\pi/3$ rad in a circle of radius 10 m.

66. A sector of a circle has a central angle of 145°. Find the area of the sector if the radius of the circle is 6 ft.

67. The area of a sector of a circle with a central angle of 140° is 70 m^2. Find the radius of the circle.

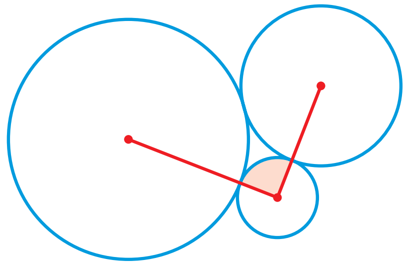
68. The area of a sector of a circle with a central angle of $5\pi/12$ rad is 20 m^2. Find the radius of the circle.

69. A sector of a circle of radius 80 mi has an area of 1600 mi^2. Find the central angle (in radians) of the sector.

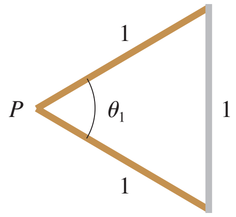
70. The area of a circle is 600 m^2. Find the area of a sector of this circle that subtends a central angle of 3 rad.

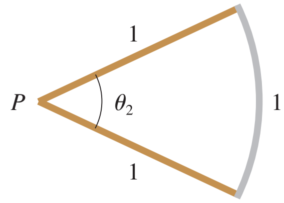
SKILLS Plus

71. Area of a Sector of a Circle Three circles with radii 1, 2, and 3 ft are externally tangent to one another, as shown in the figure. Find the area of the sector of the circle of radius 1 that is cut off by the line segments joining the center of that circle to the centers of the other two circles.



72. Comparing a Triangle and a Sector of a Circle Two wood sticks and a metal rod, each of length 1, are connected to form a triangle with angle θ_1 at the point P, as shown in the first figure below. The rod is then bent to form an arc of a circle with center P, resulting in a smaller angle θ_2 at the point P, as shown in the second figure. Find θ_1, θ_2, and $\theta_1 - \theta_2$.





73–74 ■ Clocks and Angles In 1 h the minute hand on a clock moves through a complete circle, and the hour hand moves through $\frac{1}{12}$ of a circle.

73. Through how many radians do the minute hand and the hour hand move between 1:00 P.M. and 1:45 P.M. (on the same day)?

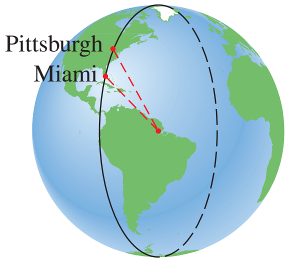
74. Through how many radians do the minute hand and the hour hand move between 1:00 P.M. and 6:45 P.M. (on the same day)?

APPLICATIONS

75. Travel Distance A car's wheels are 28 in. in diameter. How far (in mi.) will the car travel if its wheels revolve 10,000 times without slipping?

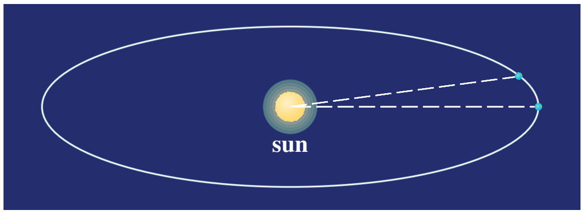
76. Wheel Revolutions How many revolutions will a car wheel of diameter 30 in. make as the car travels a distance of one mile?

77. Latitudes Pittsburgh, Pennsylvania, and Miami, Florida, lie approximately on the same meridian. Pittsburgh has a latitude of 40.5° N, and Miami has a latitude of 25.5° N. Find the distance between these two cities. (The radius of the earth is 3960 mi.)

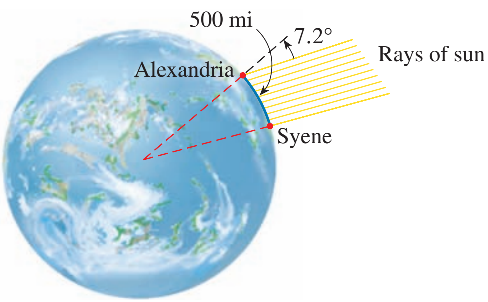


78. Latitudes Memphis, Tennessee, and New Orleans, Louisiana, lie approximately on the same meridian. Memphis has a latitude of 35° N, and New Orleans has a latitude of 30° N. Find the distance between these two cities. (The radius of the earth is 3960 mi.)

79. Orbit of the Earth Find the distance that the earth travels in one day in its path around the sun. Assume that a year has 365 days and that the path of the earth around the sun is a circle of radius 93 million miles. [*Note:* The path of the earth around the sun is actually an *ellipse* with the sun at one focus (see Section 11.2). This ellipse, however, has very small eccentricity, so it is nearly circular.]

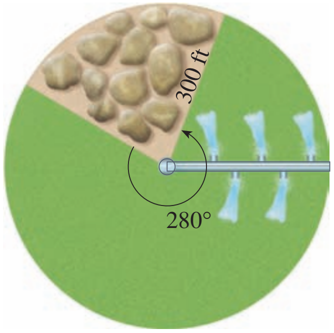


80. Circumference of the Earth The Greek mathematician Eratosthenes (ca. 276–195 B.C.) measured the circumference of the earth from the following observations. He noticed that on a certain day the sun shone directly down a deep well in Syene (modern Aswan). At the same time in Alexandria, 500 miles north (on the same meridian), the rays of the sun shone at an angle of 7.2° to the zenith. Use this information and the figure to find the radius and circumference of the earth.

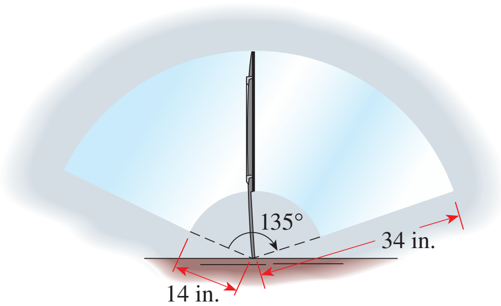


81. Nautical Miles Find the distance along an arc on the surface of the earth that subtends a central angle of 1 minute $\left(1 \text{ minute} = \frac{1}{60} \text{ degree}\right)$. This distance is called a *nautical mile*. (The radius of the earth is 3960 mi.)

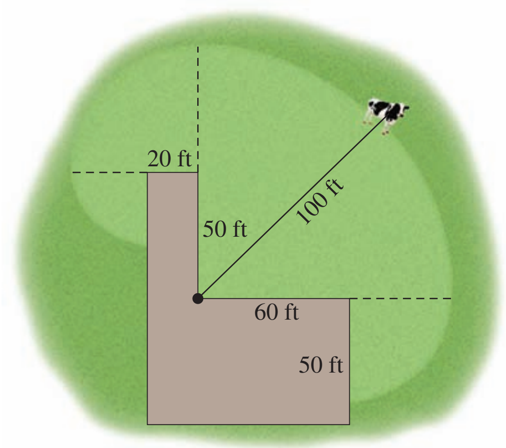
82. Irrigation An irrigation system uses a straight sprinkler pipe 300 ft long that pivots around a central point as shown. Because of an obstacle the pipe is allowed to pivot through 280° only. Find the area irrigated by this system.



83. Windshield Wipers The top and bottom ends of a windshield wiper blade are 34 in. and 14 in., respectively, from the pivot point. While in operation, the wiper sweeps through 135°. Find the area swept by the blade.



84. The Tethered Cow A cow is tethered by a 100-ft rope to the inside corner of an L-shaped building, as shown in the figure. Find the area that the cow can graze.



85. Fan A ceiling fan with 16-in. blades rotates at 45 rpm.

(a) Find the angular speed of the fan in rad/min.

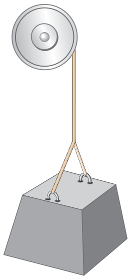
(b) Find the linear speed of the tips of the blades in in./min.

86. Radial Saw A radial saw has a blade with a 6-in. radius. Suppose that the blade spins at 1000 rpm.

(a) Find the angular speed of the blade in rad/min.

(b) Find the linear speed of the sawteeth in ft/s.

87. Winch A winch of radius 2 ft is used to lift heavy loads. If the winch makes 8 revolutions every 15 s, find the speed at which the load is rising.



88. Speed of a Car The wheels of a car have radius 11 in. and are rotating at 600 rpm. Find the speed of the car in mi/h.

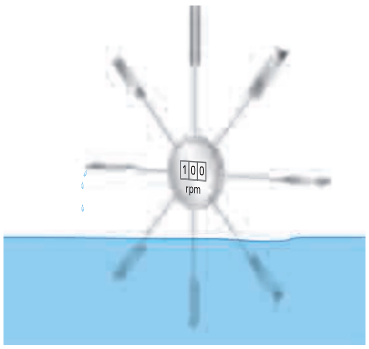
89. Speed at the Equator The earth rotates about its axis once every 23 h 56 min 4 s, and the radius of the earth is 3960 mi. Find the linear speed of a point on the equator in mi/h.

90. Truck Wheels A truck with 48-in.-diameter wheels is traveling at 50 mi/h.

(a) Find the angular speed of the wheels in rad/min.

(b) How many revolutions per minute do the wheels make?

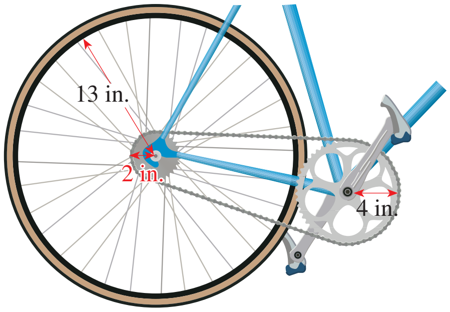
91. Speed of a Current To measure the speed of a current, scientists place a paddle wheel in the stream and observe the rate at which it rotates. If the paddle wheel has radius 0.20 m and rotates at 100 rpm, find the speed of the current in m/s.



92. Bicycle Wheel The sprockets and chain of a bicycle are shown in the figure. The pedal sprocket has a radius of 4 in., the wheel sprocket a radius of 2 in., and the wheel a radius of 13 in. The cyclist pedals at 40 rpm.

(a) Find the angular speed of the wheel sprocket.

(b) Find the speed of the bicycle. (Assume that the wheel turns at the same rate as the wheel sprocket.)



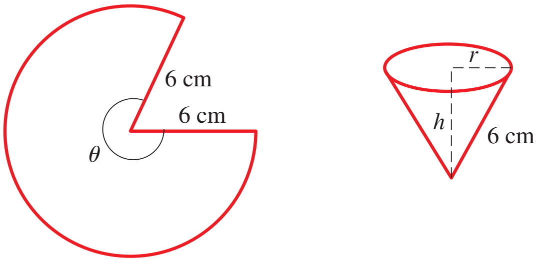
93. Conical Cup A conical cup is made from a circular piece of paper with radius 6 cm by cutting out a sector and joining the edges as shown below. Suppose $\theta = 5\pi/3$.

(a) Find the circumference C of the opening of the cup.

(b) Find the radius r of the opening of the cup. [*Hint:* Use $C = 2\pi r$.]

(c) Find the height h of the cup. [*Hint:* Use the Pythagorean Theorem.]

(d) Find the volume of the cup.



94. Conical Cup In this exercise we find the volume of the conical cup in Exercise 93 for any angle θ.

(a) Follow the steps in Exercise 93 to show that the volume of the cup as a function of θ is

$$V(\theta) = \frac{9}{\pi^2}\theta^2\sqrt{4\pi^2 - \theta^2}, \qquad 0 < \theta < 2\pi$$

(b) Graph the function V.

(c) For what angle θ is the volume of the cup a maximum?

DISCUSS ■ DISCOVER ■ PROVE ■ WRITE

95. WRITE: Different Ways of Measuring Angles The custom of measuring angles using degrees, with 360° in a circle, dates back to the ancient Babylonians, who used a number system based on groups of 60. Another system of measuring angles divides the circle into 400 units, called *grads*. In this system a right angle is 100 grad, so this fits in with our base 10 number system.

Write a short essay comparing the advantages and disadvantages of these two systems and the radian system of measuring angles. Which system do you prefer? Why?