6.3 TRIGONOMETRIC FUNCTIONS OF ANGLES

■ Trigonometric Functions of Angles ■ Evaluating Trigonometric Functions at Any Angle ■ Trigonometric Identities ■ Areas of Triangles

In Section 6.2 we defined the trigonometric ratios for acute angles. Here we extend the trigonometric ratios to all angles by defining the trigonometric functions of angles. With these functions we can solve practical problems that involve angles that are not necessarily acute.

■ Trigonometric Functions of Angles

Let POQ be a right triangle with acute angle θ as shown in Figure 1(a). Place θ in standard position as shown in Figure 1(b).

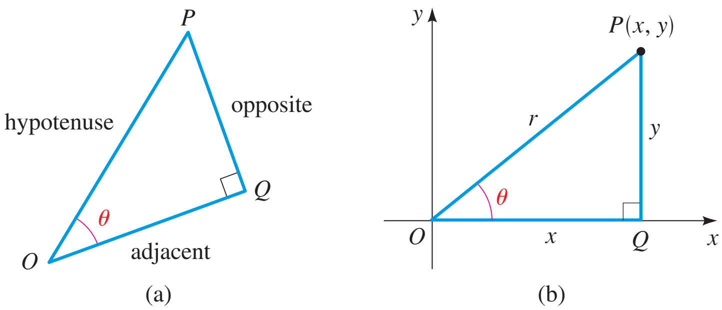


FIGURE 1

Then $P = P(x, y)$ is a point on the terminal side of θ. In triangle POQ the opposite side has length y and the adjacent side has length x. Using the Pythagorean Theorem, we see that the hypotenuse has length $r = \sqrt{x^2 + y^2}$. So

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

The other trigonometric ratios can be found in the same way.

These observations allow us to extend the trigonometric ratios to any angle. We define the trigonometric functions of angles as follows (see Figure 2).

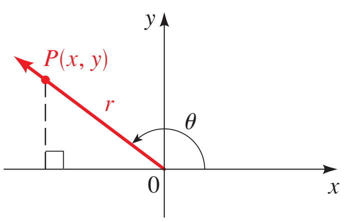


FIGURE 2

DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

Let θ be an angle in standard position, and let $P(x, y)$ be a point on the terminal side. If $r = \sqrt{x^2 + y^2}$ is the distance from the origin to the point $P(x, y)$, then

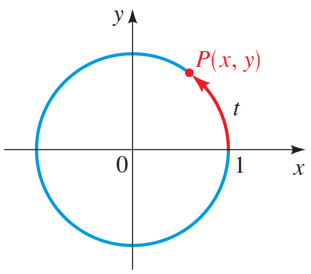
$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\csc\theta = \frac{r}{y} \quad (y \neq 0) \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

Since division by 0 is an undefined operation, certain trigonometric functions are not defined for certain angles. For example, $\tan 90° = y/x$ is undefined because $x = 0$. The angles for which the trigonometric functions may be undefined are the angles for which

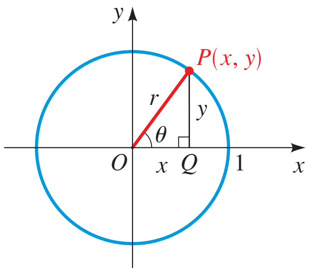
Relationship to the Trigonometric Functions of Real Numbers

You may have already studied the trigonometric functions defined by using the unit circle (Chapter 5). To see how they relate to the trigonometric functions of an *angle*, let's start with the unit circle in the coordinate plane.



$P(x, y)$ is the terminal point determined by t.

Let $P(x, y)$ be the terminal point determined by an arc of length t on the unit circle. Then t subtends an angle θ at the center of the circle. If we drop a perpendicular from P onto the point Q on the x-axis, then triangle $\triangle OPQ$ is a right triangle with legs of length x and y, as shown in the figure.



Triangle OPQ is a right triangle.

Now, by the definition of the trigonometric functions of the *real number* t we have

$$\sin t = y$$

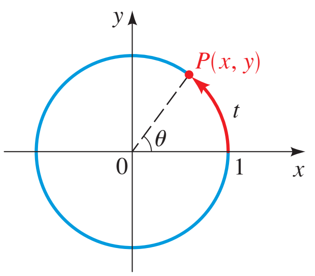
$$\cos t = x$$

By the definition of the trigonometric functions of the *angle* θ we have

$$\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{y}{1} = y$$

$$\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{x}{1} = x$$

If θ is measured in radians, then $\theta = t$. (See the figure below.) Comparing the two ways of defining the trigonometric functions, we see that they are identical. In other words, as functions they assign identical values to a given real number. (The real number is the radian measure of θ in one case or the length t of an arc in the other.)



The radian measure of angle θ is t.

Why then do we study trigonometry in two different ways? Because different applications require that we view the trigonometric functions differently. (See *Focus on Modeling*, pages 466, 533, and 581, and Sections 6.2, 6.5, and 6.6.)

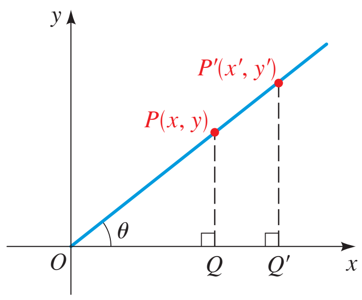
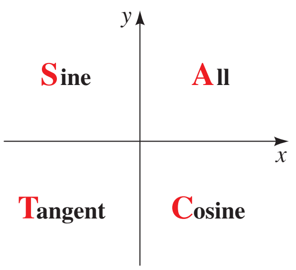


FIGURE 3

The following mnemonic device can be used to remember which trigonometric functions are positive in each quadrant: **A**ll of them, **S**ine, **T**angent, or **C**osine.



You can remember this as "**A**ll **S**tudents **T**ake **C**alculus."

either the x- or y-coordinate of a point on the terminal side of the angle is 0. These are **quadrantal angles**—angles that are coterminal with the coordinate axes.

It is a crucial fact that the values of the trigonometric functions do *not* depend on the choice of the point $P(x, y)$. This is because if $P'(x', y')$ is any other point on the terminal side, as in Figure 3, then triangles POQ and $P'OQ'$ are similar.

Evaluating Trigonometric Functions at Any Angle

From the definition we see that the values of the trigonometric functions are all positive if the angle θ has its terminal side in Quadrant I. This is because x and y are positive in this quadrant. [Of course, r is always positive, since it is simply the distance from the origin to the point $P(x, y)$.] If the terminal side of θ is in Quadrant II, however, then x is negative and y is positive. Thus in Quadrant II the functions $\sin\theta$ and $\csc\theta$ are positive, and all the other trigonometric functions have negative values. You can check the other entries in the following table.

SIGNS OF THE TRIGONOMETRIC FUNCTIONS

Quadrant	Positive Functions	Negative Functions
I	all	none
II	sin, csc	cos, sec, tan, cot
III	tan, cot	sin, csc, cos, sec
IV	cos, sec	sin, csc, tan, cot

We now turn our attention to finding the values of the trigonometric functions for angles that are not acute.

EXAMPLE 1 ■ Finding Trigonometric Functions of Angles

Find **(a)** cos 135° and **(b)** tan 390°.

SOLUTION

(a) From Figure 4 we see that $\cos 135° = -x/r$. But $\cos 45° = x/r$, and since $\cos 45° = \sqrt{2}/2$, we have

$$\cos 135° = -\frac{\sqrt{2}}{2}$$

(b) The angles 390° and 30° are coterminal. From Figure 5 it's clear that $\tan 390° = \tan 30°$, and since $\tan 30° = \sqrt{3}/3$, we have

$$\tan 390° = \frac{\sqrt{3}}{3}$$

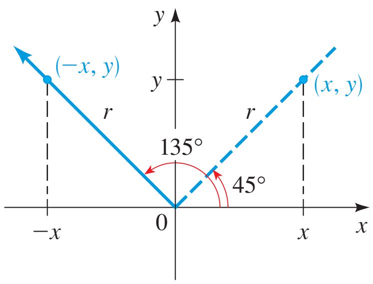


FIGURE 4

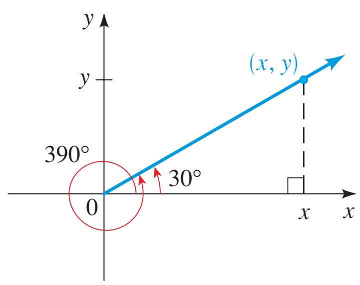


FIGURE 5

Now Try Exercises 13 and 15 ■

From Example 1 we see that the trigonometric functions for angles that aren't acute have the same value, except possibly for sign, as the corresponding trigonometric functions of an acute angle. That acute angle will be called the *reference angle*.

REFERENCE ANGLE

Let θ be an angle in standard position. The **reference angle** $\bar{\theta}$ associated with θ is the acute angle formed by the terminal side of θ and the x-axis.

Figure 6 shows that to find a reference angle $\bar{\theta}$, it's useful to know the quadrant in which the terminal side of the angle θ lies.

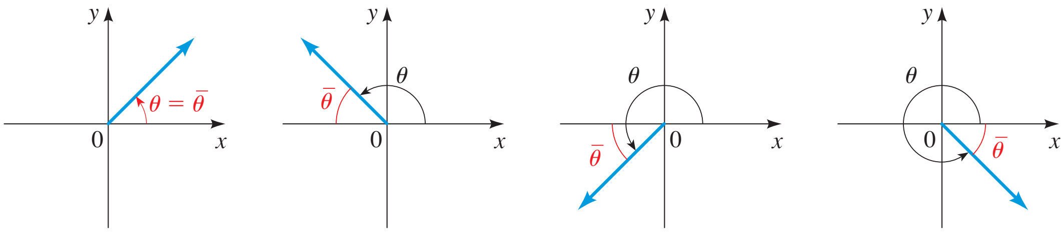


FIGURE 6 The reference angle $\bar{\theta}$ for an angle θ

EXAMPLE 2 ■ Finding Reference Angles

Find the reference angle for **(a)** $\theta = \dfrac{5\pi}{3}$ and **(b)** $\theta = 870°$.

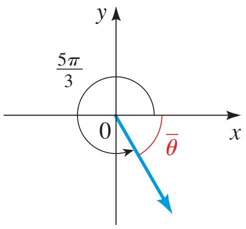


FIGURE 7

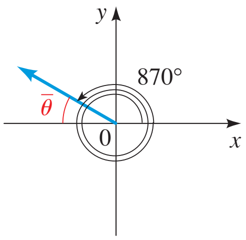


FIGURE 8

SOLUTION

(a) The reference angle is the acute angle formed by the terminal side of the angle $5\pi/3$ and the x-axis (see Figure 7). Since the terminal side of this angle is in Quadrant IV, the reference angle is

$$\bar{\theta} = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

(b) The angles 870° and 150° are coterminal [because $870 - 2(360) = 150$]. Thus the terminal side of this angle is in Quadrant II (see Figure 8). So the reference angle is

$$\bar{\theta} = 180° - 150° = 30°$$

Now Try Exercises 5 and 9 ■

EVALUATING TRIGONOMETRIC FUNCTIONS FOR ANY ANGLE

To find the values of the trigonometric functions for any angle θ, we carry out the following steps.

1. Find the reference angle $\bar{\theta}$ associated with the angle θ.
2. Determine the sign of the trigonometric function of θ by noting the quadrant in which θ lies.
3. The value of the trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\bar{\theta}$.

EXAMPLE 3 ■ Using the Reference Angle to Evaluate Trigonometric Functions

Find **(a)** sin 240° and **(b)** cot 495°.

SOLUTION

(a) This angle has its terminal side in Quadrant III, as shown in Figure 9. The reference angle is therefore 240° − 180° = 60°, and the value of sin 240° is negative. Thus

$$\sin 240° = -\sin 60° = -\frac{\sqrt{3}}{2}$$

Sign — Reference angle

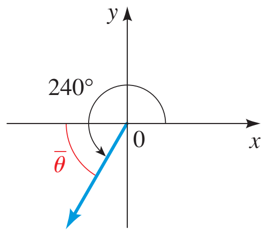


FIGURE 9

S | A
T | C

sin 240° is *negative*.

(b) The angle 495° is coterminal with the angle 135°, and the terminal side of this angle is in Quadrant II, as shown in Figure 10. So the reference angle is 180° − 135° = 45°, and the value of cot 495° is negative. We have

$$\cot 495° = \cot 135° = -\cot 45° = -1$$

Coterminal angles — Sign — Reference angle

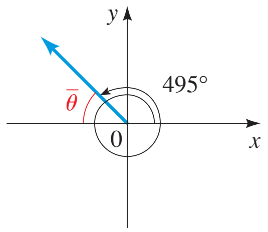


FIGURE 10

S | A
T | C

tan 495° is *negative*, so cot 495° is *negative*.

Now Try Exercises 19 and 21 ■

EXAMPLE 4 ■ Using the Reference Angle to Evaluate Trigonometric Functions

Find **(a)** $\sin \dfrac{16\pi}{3}$ and **(b)** $\sec\left(-\dfrac{\pi}{4}\right)$.

SOLUTION

(a) The angle $16\pi/3$ is coterminal with $4\pi/3$, and these angles are in Quadrant III (see Figure 11). Thus the reference angle is $(4\pi/3) - \pi = \pi/3$. Since the value of sine is negative in Quadrant III, we have

$$\sin\frac{16\pi}{3} = \sin\frac{4\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2}$$

Coterminal angles — Sign — Reference angle

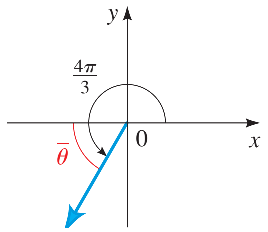


FIGURE 11

S | A
T | C

$\sin\frac{16\pi}{3}$ is *negative*.

(b) The angle $-\pi/4$ is in Quadrant IV, and its reference angle is $\pi/4$ (see Figure 12). Since secant is positive in this quadrant, we get

$$\sec\left(-\frac{\pi}{4}\right) = +\sec\frac{\pi}{4} = \sqrt{2}$$

Sign — Reference angle

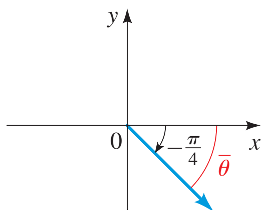


FIGURE 12

S | A
T | C

$\cos(-\frac{\pi}{4})$ is *positive*, so $\sec(-\frac{\pi}{4})$ is *positive*.

Now Try Exercises 25 and 27 ■

■ Trigonometric Identities

The trigonometric functions of angles are related to each other through several important equations called **trigonometric identities**. We've already encountered the reciprocal identities. These identities continue to hold for any angle θ, provided that both sides of the equation are defined. The Pythagorean identities are a consequence of the Pythagorean Theorem.*

FUNDAMENTAL IDENTITIES

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \csc^2\theta$$

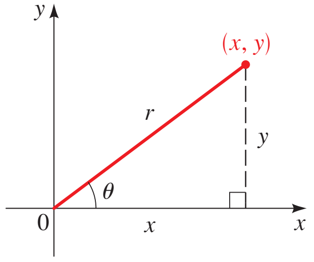


FIGURE 13

Proof Let's prove the first Pythagorean identity. Using $x^2 + y^2 = r^2$ (the Pythagorean Theorem) in Figure 13, we have

$$\sin^2\theta + \cos^2\theta = \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2 = \frac{x^2 + y^2}{r^2} = \frac{r^2}{r^2} = 1$$

Thus $\sin^2\theta + \cos^2\theta = 1$. (Although the figure indicates an acute angle, you should check that the proof holds for all angles θ.) ■

See Exercise 76 for the proofs of the other two Pythagorean identities.

EXAMPLE 5 ■ Expressing One Trigonometric Function in Terms of Another

(a) Express $\sin\theta$ in terms of $\cos\theta$.

(b) Express $\tan\theta$ in terms of $\sin\theta$, where θ is in Quadrant II.

SOLUTION

(a) From the first Pythagorean identity we get

$$\sin\theta = \pm\sqrt{1 - \cos^2\theta}$$

where the sign depends on the quadrant. If θ is in Quadrant I or II, then $\sin\theta$ is positive, so

$$\sin\theta = \sqrt{1 - \cos^2\theta}$$

whereas if θ is in Quadrant III or IV, $\sin\theta$ is negative, so

$$\sin\theta = -\sqrt{1 - \cos^2\theta}$$

(b) Since $\tan\theta = \sin\theta/\cos\theta$, we need to write $\cos\theta$ in terms of $\sin\theta$. By part (a)

$$\cos\theta = \pm\sqrt{1 - \sin^2\theta}$$

and since $\cos\theta$ is negative in Quadrant II, the negative sign applies here. Thus

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{-\sqrt{1 - \sin^2\theta}}$$

Now Try Exercise 41 ■

*We follow the usual convention of writing $\sin^2\theta$ for $(\sin\theta)^2$. In general, we write $\sin^n\theta$ for $(\sin\theta)^n$ for all integers n except $n = -1$. The superscript $n = -1$ will be assigned another meaning in Section 6.4. Of course, the same convention applies to the other five trigonometric functions.

EXAMPLE 6 ■ Evaluating a Trigonometric Function

If $\tan\theta = \frac{2}{3}$ and θ is in Quadrant III, find $\cos\theta$.

SOLUTION 1 We need to write $\cos\theta$ in terms of $\tan\theta$. From the identity $\tan^2\theta + 1 = \sec^2\theta$ we get $\sec\theta = \pm\sqrt{\tan^2\theta + 1}$. In Quadrant III, $\sec\theta$ is negative, so

$$\sec\theta = -\sqrt{\tan^2\theta + 1}$$

Thus

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{-\sqrt{\tan^2\theta + 1}}$$

$$= \frac{1}{-\sqrt{\left(\frac{2}{3}\right)^2 + 1}} = \frac{1}{-\sqrt{\frac{13}{9}}} = -\frac{3}{\sqrt{13}}$$

If you wish to rationalize the denominator, you can express $\cos\theta$ as

$$-\frac{3}{\sqrt{13}}\cdot\frac{\sqrt{13}}{\sqrt{13}} = -\frac{3\sqrt{13}}{13}$$

SOLUTION 2 This problem can be solved more easily by using the method of Example 2 of Section 6.2. Recall that, except for sign, the values of the trigonometric functions of any angle are the same as those of an acute angle (the reference angle). So, ignoring the sign for the moment, let's sketch a right triangle with an acute angle $\overline{\theta}$ satisfying $\tan\overline{\theta} = \frac{2}{3}$ (see Figure 14). By the Pythagorean Theorem the hypotenuse of this triangle has length $\sqrt{13}$. From the triangle in Figure 14 we immediately see that $\cos\overline{\theta} = 3/\sqrt{13}$. Since θ is in Quadrant III, $\cos\theta$ is negative, so

$$\cos\theta = -\frac{3}{\sqrt{13}}$$

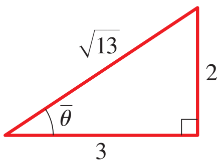


FIGURE 14

Now Try Exercise 47 ■

EXAMPLE 7 ■ Evaluating Trigonometric Functions

If $\sec\theta = 2$ and θ is in Quadrant IV, find the other five trigonometric functions of θ.

SOLUTION We sketch a triangle as in Figure 15 so that $\sec\overline{\theta} = 2$. Taking into account the fact that θ is in Quadrant IV, we get

$$\sin\theta = -\frac{\sqrt{3}}{2} \qquad \cos\theta = \frac{1}{2} \qquad \tan\theta = -\sqrt{3}$$

$$\csc\theta = -\frac{2}{\sqrt{3}} \qquad \sec\theta = 2 \qquad \cot\theta = -\frac{1}{\sqrt{3}}$$

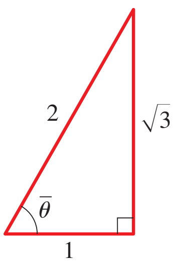


FIGURE 15

Now Try Exercise 49 ■

Areas of Triangles

We conclude this section with an application of the trigonometric functions that involves angles that are not necessarily acute. More extensive applications appear in Sections 6.5 and 6.6.

The area of a triangle is $\mathcal{A} = \frac{1}{2} \times \text{base} \times \text{height}$. If we know two sides and the included angle of a triangle, then we can find the height using the trigonometric functions, and from this we can find the area.

If θ is an acute angle, then the height of the triangle in Figure 16(a) is given by $h = b\sin\theta$. Thus the area is

$$\mathcal{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab\sin\theta$$

If the angle θ is not acute, then from Figure 16(b) we see that the height of the triangle is

$$h = b\sin(180^\circ - \theta) = b\sin\theta$$

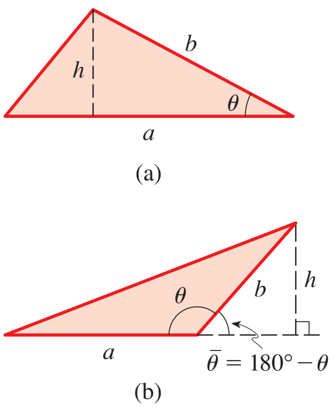


FIGURE 16

This is so because the reference angle of θ is the angle $180° - \theta$. Thus in this case also the area of the triangle is

$$\mathcal{A} = \tfrac{1}{2} \times \text{base} \times \text{height} = \tfrac{1}{2}ab \sin\theta$$

AREA OF A TRIANGLE

The area $\mathcal{A}$ of a triangle with sides of lengths a and b and with included angle θ is

$$\mathcal{A} = \tfrac{1}{2}ab \sin\theta$$

EXAMPLE 8 ■ Finding the Area of a Triangle

Find the area of triangle ABC shown in Figure 17.

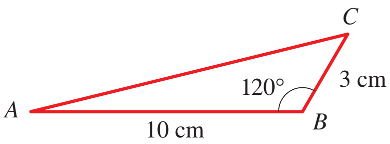


FIGURE 17

SOLUTION The triangle has sides of length 10 cm and 3 cm, with included angle 120°. Therefore

$$\begin{aligned}\mathcal{A} &= \tfrac{1}{2}ab \sin\theta \\ &= \tfrac{1}{2}(10)(3) \sin 120° \\ &= 15 \sin 60° && \text{Reference angle} \\ &= 15\frac{\sqrt{3}}{2} \approx 13 \text{ cm}^2\end{aligned}$$

Now Try Exercise 57 ■

6.3 EXERCISES

CONCEPTS

1. If the angle θ is in standard position and $P(x, y)$ is a point on the terminal side of θ, and r is the distance from the origin to P, then

   $\sin\theta = \frac{\_\_\_}{\_\_\_}$ $\cos\theta = \frac{\_\_\_}{\_\_\_}$ $\tan\theta = \frac{\_\_\_}{\_\_\_}$

2. The sign of a trigonometric function of θ depends on the ________ in which the terminal side of the angle θ lies.
   In Quadrant II, $\sin\theta$ is ________ (positive / negative).
   In Quadrant III, $\cos\theta$ is ________ (positive / negative).
   In Quadrant IV, $\sin\theta$ is ________ (positive / negative).

3. **(a)** If θ is in standard position, then the reference angle $\bar{\theta}$ is the acute angle formed by the terminal side of θ and the ________. So the reference angle for $\theta = 100°$ is $\bar{\theta} =$ ________, and that for $\theta = 190°$ is $\bar{\theta} =$ ________.
   (b) If θ is any angle, the value of a trigonometric function of θ is the same, except possibly for sign, as the value of the trigonometric function of $\bar{\theta}$. So $\sin 100° = \sin$ ____, and $\sin 190° = -\sin$ ____.

4. The area $\mathcal{A}$ of a triangle with sides of lengths a and b and with included angle θ is given by the formula $\mathcal{A} =$ ________. So the area of the triangle with sides 4 and 7 and included angle $\theta = 30°$ is ________.

SKILLS

5–12 ■ Reference Angle Find the reference angle for the given angle.

5. **(a)** 120° **(b)** 200° **(c)** 285°
6. **(a)** 175° **(b)** 310° **(c)** 730°
7. **(a)** 225° **(b)** 810° **(c)** $-105°$
8. **(a)** 99° **(b)** $-199°$ **(c)** 359°
9. **(a)** $\frac{7\pi}{10}$ **(b)** $\frac{9\pi}{8}$ **(c)** $\frac{10\pi}{3}$
10. **(a)** $\frac{5\pi}{6}$ **(b)** $\frac{10\pi}{9}$ **(c)** $\frac{23\pi}{7}$
11. **(a)** $\frac{5\pi}{7}$ **(b)** -1.4π **(c)** 1.4
12. **(a)** 2.3π **(b)** 2.3 **(c)** -10π

13–36 ■ Values of Trigonometric Functions Find the exact value of the trigonometric function.

13. $\cos 150°$ **14.** $\sin 240°$ **15.** $\tan 330°$

16. $\sin(-30°)$ **17.** $\cot(-120°)$ **18.** $\csc 300°$

19. $\csc(-630°)$ **20.** $\cot 210°$ **21.** $\cos 570°$

22. $\sec 120°$ **23.** $\tan 750°$ **24.** $\cos 660°$

25. $\sin \dfrac{3\pi}{2}$ **26.** $\cos \dfrac{4\pi}{3}$ **27.** $\tan\left(-\dfrac{4\pi}{3}\right)$

28. $\cos\left(-\dfrac{11\pi}{6}\right)$ **29.** $\csc\left(-\dfrac{5\pi}{6}\right)$ **30.** $\sec \dfrac{7\pi}{6}$

31. $\sec \dfrac{17\pi}{3}$ **32.** $\csc \dfrac{5\pi}{4}$ **33.** $\cot\left(-\dfrac{\pi}{4}\right)$

34. $\cos \dfrac{7\pi}{4}$ **35.** $\tan \dfrac{5\pi}{2}$ **36.** $\sin \dfrac{11\pi}{6}$

37–40 ■ Quadrant in Which an Angle Lies Find the quadrant in which θ lies from the information given.

37. $\sin \theta < 0$ and $\cos \theta < 0$

38. $\tan \theta < 0$ and $\sin \theta < 0$

39. $\sec \theta > 0$ and $\tan \theta < 0$

40. $\csc \theta > 0$ and $\cos \theta < 0$

41–46 ■ Expressing One Trigonometric Function in Terms of Another Write the first trigonometric function in terms of the second for θ in the given quadrant.

41. $\tan \theta$, $\cos \theta$; θ in Quadrant III

42. $\cot \theta$, $\sin \theta$; θ in Quadrant II

43. $\cos \theta$, $\sin \theta$; θ in Quadrant IV

44. $\sec \theta$, $\sin \theta$; θ in Quadrant I

45. $\sec \theta$, $\tan \theta$; θ in Quadrant II

46. $\csc \theta$, $\cot \theta$; θ in Quadrant III

47–54 ■ Values of Trigonometric Functions Find the values of the trigonometric functions of θ from the information given.

47. $\sin \theta = -\frac{4}{5}$, θ in Quadrant IV

48. $\tan \theta = \frac{4}{3}$, θ in Quadrant III

49. $\cos \theta = \frac{7}{12}$, $\sin \theta < 0$

50. $\cot \theta = -\frac{8}{9}$, $\cos \theta > 0$

51. $\csc \theta = 2$, θ in Quadrant I

52. $\cot \theta = \frac{1}{4}$, $\sin \theta < 0$

53. $\cos \theta = -\frac{2}{7}$, $\tan \theta < 0$

54. $\tan \theta = -4$, $\sin \theta > 0$

55–56 ■ Values of an Expression If $\theta = \pi/3$, find the value of each expression.

55. $\sin 2\theta$, $2 \sin \theta$ **56.** $\sin^2\theta$, $\sin(\theta^2)$

57–60 ■ Area of a Triangle Find the area of the triangle with the given description.

57. A triangle with sides of length 7 and 9 and included angle 72°

58. A triangle with sides of length 10 and 22 and included angle 10°

59. An equilateral triangle with side of length 10

60. An equilateral triangle with side of length 13

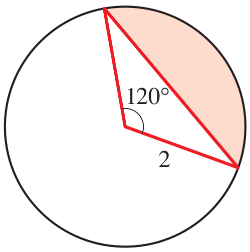
61. Finding an Angle of a Triangle A triangle has an area of 16 in^2, and two of the sides have lengths 5 in. and 7 in. Find the sine of the angle included by these two sides.

62. Finding a Side of a Triangle An isosceles triangle has an area of 24 cm^2, and the angle between the two equal sides is $5\pi/6$. Find the length of the two equal sides.

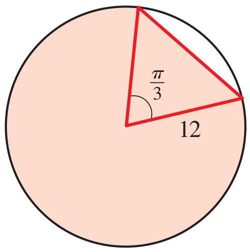
SKILLS Plus

63–64 ■ Area of a Region Find the area of the shaded region in the figure.

63.



64.



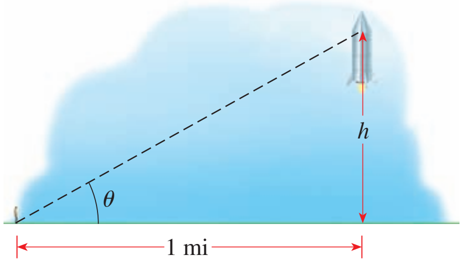
APPLICATIONS

65. Height of a Rocket A rocket fired straight up is tracked by an observer on the ground 1 mi away.

(a) Show that when the angle of elevation is θ, the height of the rocket (in ft) is $h = 5280 \tan \theta$.

(b) Complete the table to find the height of the rocket at the given angles of elevation.

θ	20°	60°	80°	85°
h				



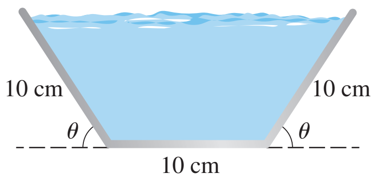
66. Rain Gutter A rain gutter is to be constructed from a metal sheet of width 30 cm by bending up one-third of the sheet on each side through an angle θ. (See the figure on the next page.)

(a) Show that the cross-sectional area of the gutter is modeled by the function

$$A(\theta) = 100 \sin \theta + 100 \sin \theta \cos \theta$$

(b) Graph the function A for $0 \le \theta \le \pi/2$.

(c) For what angle θ is the largest cross-sectional area achieved?

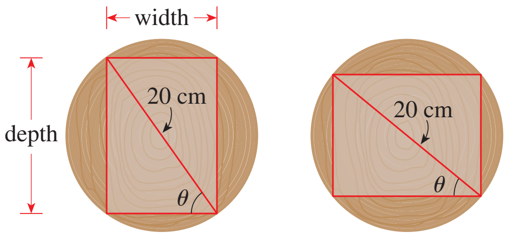


67. Wooden Beam A rectangular beam is to be cut from a cylindrical log of diameter 20 cm. The figures show different ways this can be done.

(a) Express the cross-sectional area of the beam as a function of the angle θ in the figures.

(b) Graph the function you found in part (a).

(c) Find the dimensions of the beam with largest cross-sectional area.



68. Strength of a Beam The strength of a beam is proportional to the width and the square of the depth. A beam is cut from a log as in Exercise 67. Express the strength of the beam as a function of the angle θ in the figures.

69. Throwing a Shot Put The range R and height H of a shot put thrown with an initial velocity of v_0 ft/s at an angle θ are given by

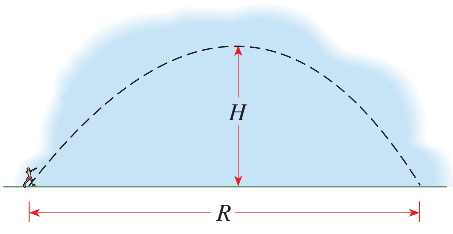
$$R = \frac{v_0^2 \sin(2\theta)}{g}$$

$$H = \frac{v_0^2 \sin^2\theta}{2g}$$

On the earth $g = 32$ ft/s^2, and on the moon $g = 5.2$ ft/s^2. Find the range and height of a shot put thrown under the given conditions.

(a) On the earth with $v_0 = 12$ ft/s and $\theta = \pi/6$

(b) On the moon with $v_0 = 12$ ft/s and $\theta = \pi/6$



70. Sledding The time in seconds that it takes for a sled to slide down a hillside inclined at an angle θ is

$$t = \sqrt{\frac{d}{16 \sin\theta}}$$

where d is the length of the slope in feet. Find the time it takes to slide down a 2000-ft slope inclined at 30°.

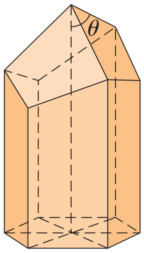
71. Beehives In a beehive each cell is a regular hexagonal prism, as shown in the figure. The amount of wax W in the cell depends on the apex angle θ and is given by

$$W = 3.02 - 0.38 \cot\theta + 0.65 \csc\theta$$

Bees instinctively choose θ so as to use the least amount of wax possible.

(a) Use a graphing device to graph W as a function of θ for $0 < \theta < \pi$.

(b) For what value of θ does W have its minimum value? [*Note:* Biologists have discovered that bees rarely deviate from this value by more than a degree or two.]



72. Turning a Corner A steel pipe is being carried down a hallway that is 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide.

(a) Show that the length of the pipe in the figure is modeled by the function

$$L(\theta) = 9 \csc\theta + 6 \sec\theta$$

(b) Graph the function L for $0 < \theta < \pi/2$.

(c) Find the minimum value of the function L.

(d) Explain why the value of L you found in part (c) is the length of the longest pipe that can be carried around the corner.