EXAMPLE 7 Evaluate $\int x^2 e^{x^3}\,dx$.

SOLUTION We substitute $u = x^3$. Then $du = 3x^2\,dx$, so $x^2\,dx = \frac{1}{3}\,du$ and

$$\int x^2 e^{x^3}\,dx = \tfrac{1}{3}\int e^u\,du = \tfrac{1}{3}e^u + C = \tfrac{1}{3}e^{x^3} + C$$

EXAMPLE 8 Find the area under the curve $y = e^{-3x}$ from 0 to 1.

SOLUTION The area is

$$A = \int_0^1 e^{-3x}\,dx = -\tfrac{1}{3}e^{-3x}\Big]_0^1 = \tfrac{1}{3}(1 - e^{-3})$$

5.3 EXERCISES

1. (a) How is the number e defined?
(b) What is an approximate value for e?
(c) Sketch, by hand, the graph of the function $f(x) = e^x$ with particular attention to how the graph crosses the y-axis. What fact allows you to do this?

2–4 ▪ Simplify each expression.

2. (a) $e^{\ln 6}$ (b) $\ln \sqrt{e}$

3. (a) $\ln e^{\sqrt{2}}$ (b) $e^{3\ln 2}$

4. (a) $\ln e^{\sin x}$ (b) $e^{x + \ln x}$

5–8 ▪ Solve each equation for x.

5. (a) $2\ln x = 1$ (b) $e^{-x} = 5$

6. (a) $e^{2x+3} - 7 = 0$ (b) $\ln(5 - 2x) = -3$

7. (a) $2^{x-5} = 3$ (b) $\ln x + \ln(x - 1) = 1$

8. (a) $\ln(\ln x) = 1$ (b) $e^{ax} = Ce^{bx}$, where $a \neq b$

9–10 ▪ Solve each inequality for x.

9. (a) $e^x < 10$ (b) $\ln x > -1$

10. (a) $2 < \ln x < 9$ (b) $e^{2-3x} > 4$

11–14 ▪ Make a rough sketch of the graph of each function. Do not use a calculator. Just use the graph given in Figure 2 and, if necessary, the transformations of Section 1.3.

11. $y = e^{-x}$

12. $y = 1 + 2e^x$

13. $y = 3 - e^x$

14. $y = 2 + 5(1 - e^{-x})$

15–20 ▪ Find the limit.

15. $\lim_{x\to\infty} e^{1-x^3}$

16. $\lim_{x\to(\pi/2)^+} e^{\tan x}$

17. $\lim_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

18. $\lim_{x\to-\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}}$

19. $\lim_{x\to 2^+} e^{3/(2-x)}$

20. $\lim_{x\to 2^-} e^{3/(2-x)}$

21–34 ▪ Differentiate the function.

21. $f(x) = x^2 e^x$

22. $y = \dfrac{e^x}{1 + x}$

23. $y = e^{ax^3}$

24. $y = e^u(\cos u + cu)$

25. $f(u) = e^{1/u}$

26. $y = e^x \ln x$

27. $F(t) = e^{t\sin 2t}$

28. $y = e^{k\tan\sqrt{x}}$

29. $y = \sqrt{1 + 2e^{3x}}$

30. $y = \cos(e^{\pi x})$

31. $y = e^{e^x}$

32. $y = \sqrt{1 + xe^{-2x}}$

33. $y = \dfrac{ae^x + b}{ce^x + d}$

34. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

35–36 ▪ Find an equation of the tangent line to the curve at the given point.

35. $y = e^{2x}\cos \pi x$, $(0, 1)$

36. $y = e^x/x$, $(1, e)$

37. Find y' if $e^{x^2 y} = x + y$.

38. Show that the function $y = Ae^{-x} + Bxe^{-x}$ satisfies the differential equation $y'' + 2y' + y = 0$.

39. For what values of r does the function $y = e^{rx}$ satisfy the equation $y'' + 6y' + 8y = 0$?

40. Find the values of λ for which $y = e^{\lambda x}$ satisfies the equation $y + y' = y''$.

41. If $f(x) = e^{2x}$, find a formula for $f^{(n)}(x)$.

42. Find the thousandth derivative of $f(x) = xe^{-x}$.

43. (a) Use the Intermediate Value Theorem to show that there is a root of the equation $e^x + x = 0$.
(b) Use Newton's method to find the root of the equation in part (a) correct to six decimal places.

44. Use a graph to find an initial approximation (to one decimal place) to the root of the equation

$$e^{-x^2} = x^3 + x - 3$$

Then use Newton's method to find the root correct to six decimal places.

45. Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where $p(t)$ is the proportion of the population that knows the rumor at time t and a and k are positive constants.
(a) Find $\lim_{t\to\infty} p(t)$.
(b) Find the rate of spread of the rumor.
(c) Graph p for the case $a = 10$, $k = 0.5$ with t measured in hours. Use the graph to estimate how long it will take for 80% of the population to hear the rumor.

46. An object is attached to the end of a vibrating spring and its displacement from its equilibrium position is $y = 8e^{-t/2}\sin 4t$, where t is measured in seconds and y is measured in centimeters.
(a) Graph the displacement function together with the functions $y = 8e^{-t/2}$ and $y = -8e^{-t/2}$. How are these graphs related? Can you explain why?
(b) Use the graph to estimate the maximum value of the displacement. Does it occur when the graph touches the graph of $y = 8e^{-t/2}$?
(c) What is the velocity of the object when it first returns to its equilibrium position?
(d) Use the graph to estimate the time after which the displacement is no more than 2 cm from equilibrium.

47. Find the absolute maximum value of the function $f(x) = x - e^x$.

48. Find the absolute minimum value of the function $g(x) = e^x/x$, $x > 0$.

49. On what interval is the curve $y = xe^{3x}$ concave upward?

50. On what interval is the function $f(x) = x^2e^{-x}$ increasing?

51–53 ■ Discuss the curve using the guidelines of Section 3.4.

51. $y = 1/(1 + e^{-x})$

52. $y = e^{2x} - e^x$

53. $y = e^{3x} + e^{-2x}$

54–55 ■ Draw a graph of f that shows all the important aspects of the curve. Estimate the local maximum and minimum values and then use calculus to find these values exactly. Use a graph of f'' to estimate the inflection points.

54. $f(x) = e^{\cos x}$

55. $f(x) = e^{x^3 - x}$

56. The family of bell-shaped curves

$$y = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$$

occurs in probability and statistics, where it is called the *normal density function*. The constant μ is called the *mean* and the positive constant σ is called the *standard deviation*. For simplicity, let's scale the function so as to remove the factor $1/(\sigma\sqrt{2\pi})$ and let's analyze the special case where $\mu = 0$. So we study the function

$$f(x) = e^{-x^2/(2\sigma^2)}$$

(a) Find the asymptote, maximum value, and inflection points of f.
(b) What role does σ play in the shape of the curve?
(c) Illustrate by graphing four members of this family on the same screen.

57–64 ■ Evaluate the integral.

57. $\int_0^5 e^{-3x}\,dx$

58. $\int_0^1 xe^{-x^2}\,dx$

59. $\int e^x\sqrt{1 + e^x}\,dx$

60. $\int \sec^2 x\, e^{\tan x}\,dx$

61. $\int \frac{e^x + 1}{e^x}\,dx$

62. $\int \frac{e^{1/x}}{x^2}\,dx$

63. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

64. $\int e^x \sin(e^x)\,dx$

65–66 ■ Find the inverse function of f. Check your answer by graphing both f and f^{-1} on the same screen.

65. $f(x) = \ln(x + 3)$

66. $f(x) = \frac{1 + e^x}{1 - e^x}$

67. If $f(x) = 3 + x + e^x$, find $(f^{-1})'(4)$.

68. Evaluate $\lim_{x\to\pi} \frac{e^{\sin x} - 1}{x - \pi}$.

69. Prove the second law of exponents [see (7)].

70. Prove the third law of exponents [see (7)].

71. (a) Show that $e^x \geqslant 1 + x$ if $x \geqslant 0$.
[*Hint:* Show that $f(x) = e^x - (1 + x)$ is increasing for $x > 0$.]
(b) Deduce that $\frac{4}{3} \leqslant \int_0^1 e^{x^2}\,dx \leqslant e$.

72. (a) Use the inequality of Exercise 71(a) to show that, for $x \geqslant 0$,

$$e^x \geqslant 1 + x + \tfrac{1}{2}x^2$$

(b) Use part (a) to improve the estimate of $\int_0^1 e^{x^2}\,dx$ given in Exercise 71(b).

73. (a) Use mathematical induction to prove that for $x \geqslant 0$ and any positive integer n,

$$e^x \geqslant 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

(b) Use part (a) to show that $e > 2.7$.
(c) Use part (a) to show that

$$\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$$

for any positive integer k.

74. This exercise illustrates Exercise 73(c) for the case $k = 10$.
(a) Compare the rates of growth of $f(x) = x^{10}$ and $g(x) = e^x$ by graphing both f and g in several viewing rectangles. When does the graph of g finally surpass the graph of f?
(b) Find a viewing rectangle that shows how the function $h(x) = e^x/x^{10}$ behaves for large x.
(c) Find a number N such that

$$\frac{e^x}{x^{10}} > 10^{10} \qquad \text{whenever} \qquad x > N$$

5.4 GENERAL LOGARITHMIC AND EXPONENTIAL FUNCTIONS

In this section we use the natural exponential and logarithmic functions to study exponential and logarithmic functions with base $a > 0$.

GENERAL EXPONENTIAL FUNCTIONS

If $a > 0$ and r is any rational number, then by (4) and (7) in Section 5.3,

$$a^r = (e^{\ln a})^r = e^{r \ln a}$$

Therefore, even for irrational numbers x, we *define*

1
$$a^x = e^{x \ln a}$$

Thus, for instance,

$$2^{\sqrt{3}} = e^{\sqrt{3}\ln 2} \approx e^{1.20} \approx 3.32$$

The function $f(x) = a^x$ is called the **exponential function with base *a***. Notice that a^x is positive for all x because e^x is positive for all x.

Definition 1 allows us to extend one of the laws of logarithms. We know that $\ln(a^r) = r \ln a$ when r is rational. But if we now let r be *any* real number we have, from Definition 1,

$$\ln a^r = \ln(e^{r \ln a}) = r \ln a$$

Thus

2
$$\ln a^r = r \ln a \qquad \text{for any real number } r$$

The general laws of exponents follow from Definition 1 together with the laws of exponents for e^x.