$$\bar{y} = \frac{1}{A}\int_0^1 \tfrac{1}{2}\{[f(x)]^2 - [g(x)]^2\}\,dx = \frac{1}{\frac{1}{6}}\int_0^1 \tfrac{1}{2}(x^2 - x^4)\,dx$$

$$= 3\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \frac{2}{5}$$

The centroid is $\left(\frac{1}{2}, \frac{2}{5}\right)$. ■

We end this section by showing a surprising connection between centroids and volumes of revolution.

▪ This theorem is named after the Greek mathematician Pappus of Alexandria, who lived in the fourth century A.D.

THEOREM OF PAPPUS Let $\mathcal{R}$ be a plane region that lies entirely on one side of a line l in the plane. If $\mathcal{R}$ is rotated about l, then the volume of the resulting solid is the product of the area A of $\mathcal{R}$ and the distance d traveled by the centroid of $\mathcal{R}$.

PROOF We give the proof for the special case in which the region lies between $y = f(x)$ and $y = g(x)$ as in Figure 16 and the line l is the y-axis. Using the method of cylindrical shells (see Section 7.3), we have

$$V = \int_a^b 2\pi x[f(x) - g(x)]\,dx = 2\pi\int_a^b x[f(x) - g(x)]\,dx$$

$$= 2\pi(\bar{x}A) \qquad \text{(by Formulas 13)}$$

$$= (2\pi\bar{x})A = Ad$$

where $d = 2\pi\bar{x}$ is the distance traveled by the centroid during one rotation about the y-axis. □

V EXAMPLE 10 A torus is formed by rotating a circle of radius r about a line in the plane of the circle that is a distance R ($> r$) from the center of the circle. Find the volume of the torus.

SOLUTION The circle has area $A = \pi r^2$. By the symmetry principle, its centroid is its center and so the distance traveled by the centroid during a rotation is $d = 2\pi R$. Therefore, by the Theorem of Pappus, the volume of the torus is

$$V = Ad = (\pi r^2)(2\pi R) = 2\pi^2 r^2 R$$ ■

The method of Example 10 should be compared with the method of Exercise 41 in Section 7.2.

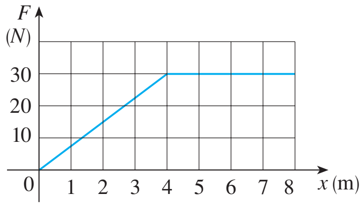
7.5 EXERCISES

1. A particle is moved along the x-axis by a force that measures $10/(1 + x)^2$ pounds at a point x feet from the origin. Find the work done in moving the particle from the origin to a distance of 9 ft.

2. When a particle is located a distance x meters from the origin, a force of $\cos(\pi x/3)$ newtons acts on it. How much work is done in moving the particle from $x = 1$ to $x = 2$? Interpret your answer by considering the work done from $x = 1$ to $x = 1.5$ and from $x = 1.5$ to $x = 2$.

3. Shown is the graph of a force function (in newtons) that increases to its maximum value and then remains constant.

How much work is done by the force in moving an object a distance of 8 m?



4. The table shows values of a force function $f(x)$ where x is measured in meters and $f(x)$ in newtons. Use Simpson's Rule to estimate the work done by the force in moving an object a distance of 18 m.

x	0	3	6	9	12	15	18
$f(x)$	9.8	9.1	8.5	8.0	7.7	7.5	7.4

5. A force of 10 lb is required to hold a spring stretched 4 in. beyond its natural length. How much work is done in stretching it from its natural length to 6 in. beyond its natural length?

6. A spring has a natural length of 20 cm. If a 25-N force is required to keep it stretched to a length of 30 cm, how much work is required to stretch it from 20 cm to 25 cm?

7. Suppose that 2 J of work is needed to stretch a spring from its natural length of 30 cm to a length of 42 cm.
(a) How much work is needed to stretch it from 35 cm to 40 cm?
(b) How far beyond its natural length will a force of 30 N keep the spring stretched?

8. If 6 J of work is needed to stretch a spring from 10 cm to 12 cm and another 10 J is needed to stretch it from 12 cm to 14 cm, what is the natural length of the spring?

9–16 ▪ Show how to approximate the required work by a Riemann sum. Then express the work as an integral and evaluate it.

9. A heavy rope, 50 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high.
(a) How much work is done in pulling the rope to the top of the building?
(b) How much work is done in pulling half the rope to the top of the building?

10. A chain lying on the ground is 10 m long and its mass is 80 kg. How much work is required to raise one end of the chain to a height of 6 m?

11. A cable that weighs 2 lb/ft is used to lift 800 lb of coal up a mine shaft 500 ft deep. Find the work done.

12. A bucket that weighs 4 lb and a rope of negligible weight are used to draw water from a well that is 80 ft deep. The bucket is filled with 40 lb of water and is pulled up at a rate of 2 ft/s, but water leaks out of a hole in the bucket at a rate of 0.2 lb/s. Find the work done in pulling the bucket to the top of the well.

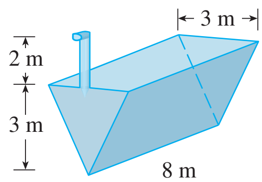
13. A leaky 10-kg bucket is lifted from the ground to a height of 12 m at a constant speed with a rope that weighs 0.8 kg/m. Initially the bucket contains 36 kg of water, but the water leaks at a constant rate and finishes draining just as the bucket reaches the 12 m level. How much work is done?

14. A 10-ft chain weighs 25 lb and hangs from a ceiling. Find the work done in lifting the lower end of the chain to the ceiling so that it's level with the upper end.

15. An aquarium 2 m long, 1 m wide, and 1 m deep is full of water. Find the work needed to pump half of the water out of the aquarium. (Use the fact that the density of water is 1000 kg/m^3.)

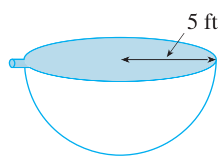
16. A circular swimming pool has a diameter of 24 ft, the sides are 5 ft high, and the depth of the water is 4 ft. How much work is required to pump all of the water out over the side? (Use the fact that water weighs 62.5 lb/ft^3.)

17. The tank shown is full of water.
(a) Find the work required to pump the water out of the spout.



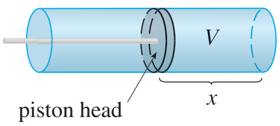
(b) Suppose that the pump breaks down after 4.7×10^5 J of work has been done. What is the depth of the water remaining in the tank?

18. The hemispherical tank shown is full of water. Given that water weighs 62.5 lb/ft^3, find the work required to pump the water out of the tank.



19. When gas expands in a cylinder with radius r, the pressure at any given time is a function of the volume: $P = P(V)$. The force exerted by the gas on the piston (see the figure) is the product of the pressure and the area: $F = \pi r^2 P$. Show that the work done by the gas when the volume expands from volume V_1 to volume V_2 is

$$W = \int_{V_1}^{V_2} P\,dV$$



20. In a steam engine the pressure P and volume V of steam satisfy the equation $PV^{1.4} = k$, where k is a constant. (This is true for adiabatic expansion, that is, expansion in which there is no heat transfer between the cylinder and its surroundings.) Use Exercise 19 to calculate the work done by the engine during a cycle when the steam starts at a pressure of 160 lb/in^2 and a volume of 100 in^3 and expands to a volume of 800 in^3.

21. (a) Newton's Law of Gravitation states that two bodies with masses m_1 and m_2 attract each other with a force

$$F = G\frac{m_1 m_2}{r^2}$$

where r is the distance between the bodies and G is the gravitational constant. If one of the bodies is fixed, find the work needed to move the other from $r = a$ to $r = b$.

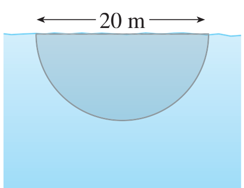
(b) Compute the work required to launch a 1000-kg satellite vertically to an orbit 1000 km high. You may assume that the Earth's mass is 5.98×10^{24} kg and is concentrated at its center. Take the radius of the Earth to be 6.37×10^6 m and $G = 6.67 \times 10^{-11}$ N·m^2/kg^2.

22. (a) Use an improper integral and information from Exercise 21 to find the work needed to propel a 1000-kg satellite out of the Earth's gravitational field.

(b) Find the *escape velocity* v_0 that is needed to propel a rocket of mass m out of the gravitational field of a planet with mass M and radius R. (Use the fact that the initial kinetic energy of $\frac{1}{2}mv_0^2$ supplies the needed work.)

23–26 ■ A vertical plate is submerged in water and has the indicated shape. Explain how to approximate the hydrostatic force against one side of the plate by a Riemann sum. Then express the force as an integral and evaluate it.

23.



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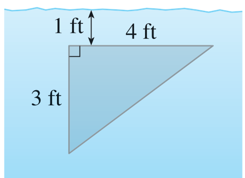

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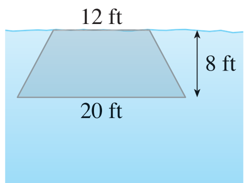

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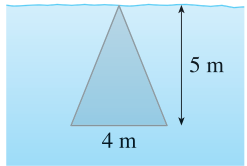
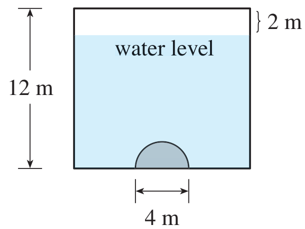


27. A trough is filled with a liquid of density 840 kg/m^3. The ends of the trough are equilateral triangles with sides 8 m long and vertex at the bottom. Find the hydrostatic force on one end of the trough.

28. A large tank is designed with ends in the shape of the region between the curves $y = \frac{1}{2}x^2$ and $y = 12$, measured in feet. Find the hydrostatic force on one end of the tank if it is filled to a depth of 8 ft with gasoline. (Assume the gasoline's density is 42.0 lb/ft^3.)

29. A swimming pool is 20 ft wide and 40 ft long and its bottom is an inclined plane, the shallow end having a depth of 3 ft and the deep end, 9 ft. If the pool is full of water, find the hydrostatic force on (a) the shallow end, (b) the deep end, (c) one of the sides, and (d) the bottom of the pool.

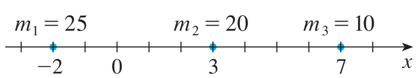
30. A vertical dam has a semicircular gate as shown in the figure. Find the hydrostatic force against the gate.



31. A vertical, irregularly shaped plate is submerged in water. The table shows measurements of its width, taken at the indicated depths. Use Simpson's Rule to estimate the force of the water against the plate.

Depth (m)	2.0	2.5	3.0	3.5	4.0	4.5	5.0
Plate width (m)	0	0.8	1.7	2.4	2.9	3.3	3.6

32. Point-masses m_i are located on the x-axis as shown. Find the moment M of the system about the origin and the center of mass $\bar{x}$.



33–34 ▪ The masses m_i are located at the points P_i. Find the moments M_x and M_y and the center of mass of the system.

33. $m_1 = 6,\ m_2 = 5,\ m_3 = 10;$
$P_1(1, 5),\ P_2(3, -2),\ P_3(-2, -1)$

34. $m_1 = 6,\ m_2 = 5,\ m_3 = 1,\ m_4 = 4;$
$P_1(1, -2),\ P_2(3, 4),\ P_3(-3, -7),\ P_4(6, -1)$

35–38 ▪ Sketch the region bounded by the curves, and visually estimate the location of the centroid. Then find the exact coordinates of the centroid.

35. $y = 4 - x^2, \quad y = 0$

36. $3x + 2y = 6, \quad y = 0, \quad x = 0$

37. $y = e^x, \quad y = 0, \quad x = 0, \quad x = 1$

38. $y = 1/x, \quad y = 0, \quad x = 1, \quad x = 2$

39–42 ▪ Find the centroid of the region bounded by the given curves.

39. $y = \sqrt{x}, \quad y = x$

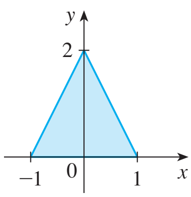
40. $y = x + 2, \quad y = x^2$

41. $y = \sin x, \quad y = \cos x, \quad x = 0, \quad x = \pi/4$

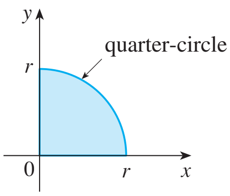
42. $y = x, \quad y = 0, \quad y = 1/x, \quad x = 2$

43–44 ▪ Calculate the moments M_x and M_y and the center of mass of a lamina with the given density and shape.

43. $\rho = 1$



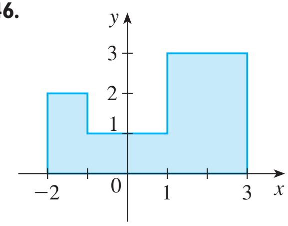
44. $\rho = 2$



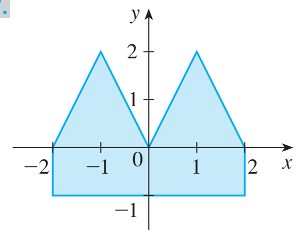
45. Prove that the centroid of any triangle is located at the point of intersection of the medians. [*Hints:* Place the axes so that the vertices are $(a, 0)$, $(0, b)$, and $(c, 0)$. Recall that a median is a line segment from a vertex to the midpoint of the opposite side. Recall also that the medians intersect at a point two-thirds of the way from each vertex (along the median) to the opposite side.]

46–47 ▪ Find the centroid of the region shown, not by integration, but by locating the centroids of the rectangles and triangles (from Exercise 45) and using additivity of moments.

46.



47.



48–50 ▪ Use the Theorem of Pappus to find the volume of the given solid.

48. A sphere of radius r (Use Example 8.)

49. A cone with height h and base radius r

50. The solid obtained by rotating the triangle with vertices $(2, 3)$, $(2, 5)$, and $(5, 4)$ about the x-axis

51. Prove Formulas 13.

52. Let $\mathcal{R}$ be the region that lies between the curves $y = x^m$ and $y = x^n$, $0 \leqslant x \leqslant 1$, where m and n are integers with $0 \leqslant n < m$.

(a) Sketch the region $\mathcal{R}$.

(b) Find the coordinates of the centroid of $\mathcal{R}$.

(c) Try to find values of m and n such that the centroid lies *outside* $\mathcal{R}$.

7.6 DIFFERENTIAL EQUATIONS

A **differential equation** is an equation that contains an unknown function and one or more of its derivatives. Here are some examples:

$$y' = xy \tag{1}$$

$$y'' + 2y' + y = 0 \tag{2}$$

$$\frac{d^3y}{dx^3} + x\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^{-x} \tag{3}$$