Then we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\cos\theta + 2\sin\theta\,\cos\theta}{-\sin\theta + \cos 2\theta} = \frac{\cos\theta + \sin 2\theta}{-\sin\theta + \cos 2\theta}$$

which is equivalent to our previous expression.

GRAPHING POLAR CURVES WITH GRAPHING DEVICES

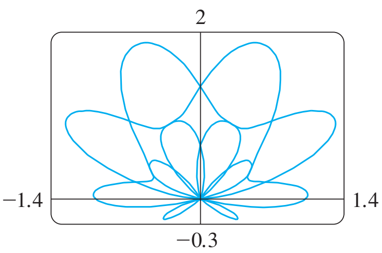


FIGURE 15
$r = \sin\theta + \sin^3(5\theta/2)$

Although it's useful to be able to sketch simple polar curves by hand, we need to use a graphing calculator or computer when we are faced with a curve as complicated as the one shown in Figure 15.

Some graphing devices have commands that enable us to graph polar curves directly. With other machines we need to convert to parametric equations first. In this case we take the polar equation $r = f(\theta)$ and write its parametric equations as

$$x = r\cos\theta = f(\theta)\cos\theta \qquad y = r\sin\theta = f(\theta)\sin\theta$$

Some machines require that the parameter be called t rather than θ.

EXAMPLE 10 Graph the curve $r = \sin(8\theta/5)$.

SOLUTION Let's assume that our graphing device doesn't have a built-in polar graphing command. In this case we need to work with the corresponding parametric equations, which are

$$x = r\cos\theta = \sin(8\theta/5)\cos\theta \qquad y = r\sin\theta = \sin(8\theta/5)\sin\theta$$

In any case we need to determine the domain for θ. So we ask ourselves: How many complete rotations are required until the curve starts to repeat itself? If the answer is n, then

$$\sin\frac{8(\theta + 2n\pi)}{5} = \sin\left(\frac{8\theta}{5} + \frac{16n\pi}{5}\right) = \sin\frac{8\theta}{5}$$

and so we require that $16n\pi/5$ be an even multiple of π. This will first occur when $n = 5$. Therefore, we will graph the entire curve if we specify that $0 \leqslant \theta \leqslant 10\pi$. Switching from θ to t, we have the equations

$$x = \sin(8t/5)\cos t \qquad y = \sin(8t/5)\sin t \qquad 0 \leqslant t \leqslant 10\pi$$

and Figure 16 shows the resulting curve. Notice that this rose has 16 loops. ■

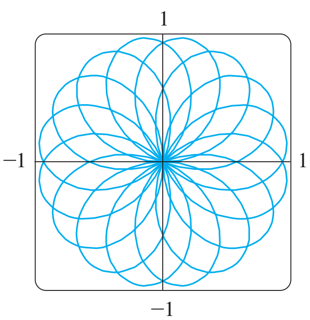


FIGURE 16
$r = \sin(8\theta/5)$

9.3 EXERCISES

1–2 ■ Plot the point whose polar coordinates are given. Then find two other pairs of polar coordinates of this point, one with $r > 0$ and one with $r < 0$.

1. (a) $(1, \pi/2)$ (b) $(-2, \pi/4)$ (c) $(3, 2)$

2. (a) $(3, 0)$ (b) $(2, -\pi/7)$ (c) $(-1, -\pi/2)$

3–4 ■ Plot the point whose polar coordinates are given. Then find the Cartesian coordinates of the point.

3. (a) $(3, \pi/2)$ (b) $(2\sqrt{2}, 3\pi/4)$ (c) $(-1, \pi/3)$

4. (a) $(2, 2\pi/3)$ (b) $(4, 3\pi)$ (c) $(-2, -5\pi/6)$

5–6 ▪ The Cartesian coordinates of a point are given.

(i) Find polar coordinates (r, θ) of the point, where $r > 0$ and $0 \leqslant \theta < 2\pi$.

(ii) Find polar coordinates (r, θ) of the point, where $r < 0$ and $0 \leqslant \theta < 2\pi$.

5. (a) $(1, 1)$ (b) $(2\sqrt{3}, -2)$

6. (a) $(-1, -\sqrt{3})$ (b) $(-2, 3)$

7–12 ▪ Sketch the region in the plane consisting of points whose polar coordinates satisfy the given conditions.

7. $1 \leqslant r \leqslant 2$

8. $r \geqslant 0, \quad \pi/3 \leqslant \theta \leqslant 2\pi/3$

9. $0 \leqslant r < 4, \quad -\pi/2 \leqslant \theta < \pi/6$

10. $2 < r \leqslant 5, \quad 3\pi/4 < \theta < 5\pi/4$

11. $2 < r < 3, \quad 5\pi/3 \leqslant \theta \leqslant 7\pi/3$

12. $-1 \leqslant r \leqslant 1, \quad \pi/4 \leqslant \theta \leqslant 3\pi/4$

13–16 ▪ Identify the curve by finding a Cartesian equation for the curve.

13. $r = 3 \sin \theta$

14. $r = 2 \sin \theta + 2 \cos \theta$

15. $r = \csc \theta$

16. $r = \tan \theta \sec \theta$

17–20 ▪ Find a polar equation for the curve represented by the given Cartesian equation.

17. $x = -y^2$

18. $x + y = 9$

19. $x^2 + y^2 = 2cx$

20. $x^2 - y^2 = 1$

21–22 ▪ For each of the described curves, decide if the curve would be more easily given by a polar equation or a Cartesian equation. Then write an equation for the curve.

21. (a) A line through the origin that makes an angle of $\pi/6$ with the positive x-axis
(b) A vertical line through the point $(3, 3)$

22. (a) A circle with radius 5 and center $(2, 3)$
(b) A circle centered at the origin with radius 4

23–40 ▪ Sketch the curve with the given polar equation.

23. $\theta = -\pi/6$

24. $r^2 - 3r + 2 = 0$

25. $r = \sin \theta$

26. $r = -3 \cos \theta$

27. $r = 2(1 - \sin \theta), \quad \theta \geqslant 0$

28. $r = 1 - 3 \cos \theta$

29. $r = \theta, \quad \theta \geqslant 0$

30. $r = \ln \theta, \quad \theta \geqslant 1$

31. $r = \sin 2\theta$

32. $r = 2 \cos 3\theta$

33. $r = 2 \cos 4\theta$

34. $r = \sin 5\theta$

35. $r^2 = 4 \cos 2\theta$

36. $r^2 = \sin 2\theta$

37. $r = 2 \cos(3\theta/2)$

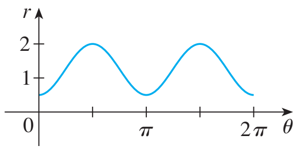
38. $r^2\theta = 1$

39. $r = 1 + 2 \cos 2\theta$

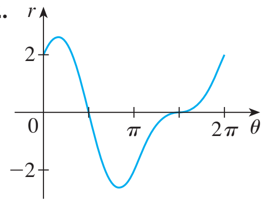
40. $r = 1 + 2 \cos(\theta/2)$

41–42 ▪ The figure shows the graph of r as a function of θ in Cartesian coordinates. Use it to sketch the corresponding polar curve.

41.



42.



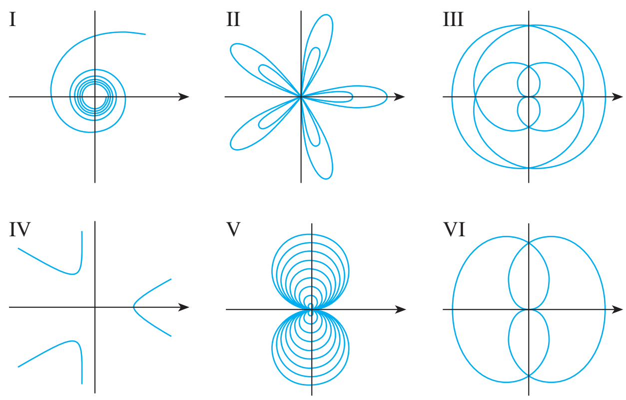
43. Show that the polar curve $r = 4 + 2 \sec \theta$ (called a **conchoid**) has the line $x = 2$ as a vertical asymptote by showing that $\lim_{r \to \pm\infty} x = 2$. Use this fact to help sketch the conchoid.

44. Sketch the curve $(x^2 + y^2)^3 = 4x^2y^2$.

45. Show that the curve $r = \sin \theta \tan \theta$ (called a **cissoid of Diocles**) has the line $x = 1$ as a vertical asymptote. Show also that the curve lies entirely within the vertical strip $0 \leqslant x < 1$. Use these facts to help sketch the cissoid.

46. Match the polar equations with the graphs labeled I–VI. Give reasons for your choices. (Don't use a graphing device.)

(a) $r = \sin(\theta/2)$ (b) $r = \sin(\theta/4)$
(c) $r = \sec(3\theta)$ (d) $r = \theta \sin \theta$
(e) $r = 1 + 4 \cos 5\theta$ (f) $r = 1/\sqrt{\theta}$



47–50 ▪ Find the slope of the tangent line to the given polar curve at the point specified by the value of θ.

47. $r = 2 \sin \theta, \quad \theta = \pi/6$

48. $r = 2 - \sin \theta, \quad \theta = \pi/3$

49. $r = 1/\theta, \quad \theta = \pi$

50. $r = \sin 3\theta, \quad \theta = \pi/6$

51–54 ▪ Find the points on the given curve where the tangent line is horizontal or vertical.

51. $r = 3\cos\theta$

52. $r = e^{\theta}$

53. $r = 1 + \cos\theta$

54. $r^2 = \sin 2\theta$

55. Show that the polar equation $r = a\sin\theta + b\cos\theta$, where $ab \neq 0$, represents a circle, and find its center and radius.

56. Show that the curves $r = a\sin\theta$ and $r = a\cos\theta$ intersect at right angles.

57–60 ▪ Use a graphing device to graph the polar curve. Choose the parameter interval to make sure that you produce the entire curve.

57. $r = e^{\sin\theta} - 2\cos(4\theta)$ (butterfly curve)

58. $r = \sin^2(4\theta) + \cos(4\theta)$

59. $r = 2 - 5\sin(\theta/6)$

60. $r = \cos(\theta/2) + \cos(\theta/3)$

61. How are the graphs of $r = 1 + \sin(\theta - \pi/6)$ and $r = 1 + \sin(\theta - \pi/3)$ related to the graph of $r = 1 + \sin\theta$? In general, how is the graph of $r = f(\theta - \alpha)$ related to the graph of $r = f(\theta)$?

62. Use a graph to estimate the y-coordinate of the highest points on the curve $r = \sin 2\theta$. Then use calculus to find the exact value.

63. (a) Investigate the family of curves defined by the polar equations $r = \sin n\theta$, where n is a positive integer. How is the number of loops related to n?
(b) What happens if the equation in part (a) is replaced by $r = |\sin n\theta|$?

64. A family of curves is given by the equations $r = 1 + c\sin n\theta$, where c is a real number and n is a positive integer. How does the graph change as n increases? How does it change as c changes? Illustrate by graphing enough members of the family to support your conclusions.

65. A family of curves has polar equations

$$r = \frac{1 - a\cos\theta}{1 + a\cos\theta}$$

Investigate how the graph changes as the number a changes. In particular, you should identify the transitional values of a for which the basic shape of the curve changes.

66. The astronomer Giovanni Cassini (1625–1712) studied the family of curves with polar equations

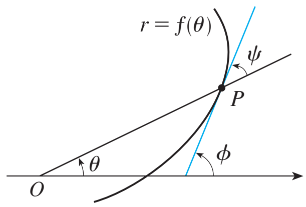
$$r^4 - 2c^2r^2\cos 2\theta + c^4 - a^4 = 0$$

where a and c are positive real numbers. These curves are called the **ovals of Cassini** even though they are oval shaped only for certain values of a and c. (Cassini thought that these curves might represent planetary orbits better than Kepler's ellipses.) Investigate the variety of shapes that these curves may have. In particular, how are a and c related to each other when the curve splits into two parts?

67. Let P be any point (except the origin) on the curve $r = f(\theta)$. If ψ is the angle between the tangent line at P and the radial line OP, show that

$$\tan\psi = \frac{r}{dr/d\theta}$$

[*Hint:* Observe that $\psi = \phi - \theta$ in the figure.]



68. (a) Use Exercise 67 to show that the angle between the tangent line and the radial line is $\psi = \pi/4$ at every point on the curve $r = e^{\theta}$.
(b) Illustrate part (a) by graphing the curve and the tangent lines at the points where $\theta = 0$ and $\pi/2$.
(c) Prove that any polar curve $r = f(\theta)$ with the property that the angle ψ between the radial line and the tangent line is a constant must be of the form $r = Ce^{k\theta}$, where C and k are constants.

9.4 AREAS AND LENGTHS IN POLAR COORDINATES

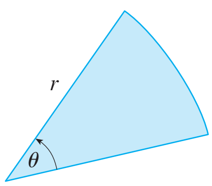


FIGURE 1

In this section we develop the formula for the area of a region whose boundary is given by a polar equation. We need to use the formula for the area of a sector of a circle

1 $$A = \tfrac{1}{2}r^2\theta$$

where, as in Figure 1, r is the radius and θ is the radian measure of the central angle.