MATH 202

§7.4

[1] $f(x) = 2 - 3x \longrightarrow f'(x) = -3$

$$L = \int_{-2}^{1} \sqrt{1 + (-3)^2}\, dx = \sqrt{10} \int_{-2}^{1} dx = \boxed{3\sqrt{10}}$$

CHECK LET $P_1 = (-2, f(-2)) = (-2, 8)$

$P_2 = (1, f(1)) = (1, -1)$

THEN DISTANCE FROM P_1 TO P_2 $= \sqrt{(-2-1)^2 + (8+1)^2} = \sqrt{9 + 81} = \sqrt{90} = \boxed{3\sqrt{10}}$ ✓

[3] $f(x) = 1 + 6x^{3/2}$

$f'(x) = 9x^{1/2}$

$$L = \int_0^1 \sqrt{1 + (9x^{1/2})^2}\, dx = \int_0^1 \sqrt{1 + 81x}\, dx$$

LET $u = 1 + 81x$ | $x = 0 \Rightarrow u = 1$ | $\longrightarrow \int_1^{82} \frac{1}{81}\sqrt{u}\, du$

$du = 81\, dx$ | $x = 1 \Rightarrow u = 82$ |

$$= \frac{2}{243} u^{3/2} \Big|_1^{82} = \boxed{\frac{2}{243}\left(82^{3/2} - 1\right)}$$

[4] $y^2 = 4(x+4)^3 \longrightarrow y = 2(x+4)^{3/2}$ $(y > 0)$

So $f(x) = 2(x+4)^{3/2}$

$f'(x) = 3\sqrt{x+4}$

$$L = \int_0^2 \sqrt{1 + 9(x+4)}\, dx = \int_0^2 \sqrt{9x + 37}\, dx$$

LET $u = 9x + 37$ | $x = 0 \Rightarrow u = 37$ | $\longrightarrow \int_{37}^{55} \frac{1}{9}\sqrt{u}\, du$

$du = 9\, dx$ | $x = 2 \Rightarrow u = 55$ |

$$= \frac{2}{27} u^{3/2} \Big|_{37}^{55} = \boxed{\frac{2}{27}\left(55^{3/2} - 37^{3/2}\right)}$$

5 $f(x) = \frac{x^5}{6} + \frac{1}{10x^3} \longrightarrow f'(x) = \frac{5}{6}x^4 - \frac{3}{10}x^{-4}$

$$1 + f'(x)^2 = 1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100}x^{-8} = \frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100}x^{-8}$$

$$= \left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2$$

$$\therefore L = \int_1^2 \sqrt{1 + f'(x)^2}\, dx = \int_1^2 \sqrt{\left(\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right)^2}\, dx$$

$$= \int_1^2 \left|\frac{5}{6}x^4 + \frac{3}{10}x^{-4}\right| dx$$

NOTE: ABS. VAL. SIGN UNNECESSARY SINCE $x > 0$

$$= \left[\frac{1}{6}x^5 - \frac{1}{10}x^{-3}\right]_1^2 = \left(\frac{32}{6} - \frac{1}{80}\right) - \left(\frac{1}{6} - \frac{1}{10}\right) = \boxed{\frac{1261}{240}}$$

6 $f(x) = \frac{x^2}{2} - \frac{1}{4}\ln x \longrightarrow f'(x) = x - \frac{1}{4}x^{-1}$

$$1 + f'(x)^2 = 1 + x^2 - \frac{1}{2} + \frac{1}{16}x^{-2} = x^2 + \frac{1}{2} + \frac{1}{16}x^{-2} = \left(x + \frac{1}{4}x^{-1}\right)^2$$

$$\therefore L = \int_2^4 \sqrt{1 + f'(x)^2}\, dx = \int_1^4 \sqrt{\left(x + \frac{1}{4}x^{-1}\right)^2} = \int_1^4 x + \frac{1}{4}x^{-1}\, dx$$

$$= \left[\frac{1}{2}x^2 + \frac{1}{4}\ln x\right]_1^4 = \left(8 + \frac{1}{4}\ln(4)\right) - \frac{1}{2} = \boxed{\frac{1}{4}\left(30 + \ln(4)\right)}$$

7 $f(y) = \frac{1}{3}\sqrt{y}\,(y-3)$

$$f'(y) = \frac{1}{3}\left[\frac{1}{2\sqrt{y}}(y-3) + \sqrt{y}\right] = \frac{1}{3}\left(\frac{1}{2}y^{1/2} - \frac{3}{2}y^{-1/2} + y^{1/2}\right)$$

$$= \frac{1}{2}\left(y^{1/2} - y^{-1/2}\right)$$

$$1 + f'(y)^2 = 1 + \frac{1}{4}y - \frac{1}{2} + \frac{1}{4}y^{-1} = \left(\frac{1}{2}\left(y^{1/2} + y^{-1/2}\right)\right)^2$$

$$\therefore\ L = \int_1^9 \frac{1}{2}\left(y^{1/2} + y^{-1/2}\right)dy = \frac{1}{2}\left[\frac{2}{3}y^{3/2} + 2y^{1/2}\right]_1^9$$

$$= \frac{1}{2}\left[(18+6) - \left(\frac{2}{3} + 2\right)\right] = \boxed{\frac{32}{3}}$$

9 $f(x) = \ln(\sec x)$

$$f'(x) = \frac{1}{\sec x}\tan x \sec x = \tan x$$

$$1 + f'(x)^2 = 1 + \tan^2 x = \sec^2 x$$

Note: $\sec x > 0$ for $0 \le x \le \pi/4$

$$\therefore\ L = \int_0^{\pi/4} \sqrt{\sec^2 x}\,dx = \int_0^{\pi/4} \sec x\,dx = \ln|\sec x + \tan x|\Big|_0^{\pi/4}$$

$$= \ln|\sqrt{2} + 1| - \ln|1 + 0| = \boxed{\ln|\sqrt{2} + 1|}$$

11 $f(x) = \cosh x$

$f'(x) = \sinh x$

$1 + f'(x)^2 = 1 + \sinh^2 x = \cosh^2 x$ POSITIVE FOR ALL x

$\therefore L = \int_0^1 \sqrt{\cosh^2 x}\, dx = \int_0^1 \cosh x \, dx = \sinh x \Big|_0^1 = \sinh(1)$

$= \boxed{\frac{1}{2}\left(e - \frac{1}{e}\right)}$

12 $y^2 = 4x \Rightarrow x = f(y) = \frac{1}{4} y^2$

$f'(y) = \frac{1}{2} y$

$1 + f'(y)^2 = 1 + \left(\frac{1}{2} y\right)^2$

$\therefore L = \int_0^2 \sqrt{1 + \left(\frac{1}{2} y\right)^2}\, dy$

LET $\frac{1}{2} y = \tan\theta$

$y = 2\tan\theta$

$dy = 2\sec^2\theta \, d\theta$

$y = 0 \Rightarrow \theta = 0$

$y = 2 \Rightarrow \theta = \frac{\pi}{4}$

$2 \int_0^{\pi/4} \sqrt{1 + \tan^2\theta}\, \sec^2\theta \, d\theta = 2 \int_0^{\pi/4} \sec^3\theta \, d\theta$ (THIS INTEGRAL WAS DONE IN CLASS!! TODAY!!)

INT. BY PARTS FOLLOWED BY TRIG IDENTITIES

$= \boxed{\sqrt{2} + \ln(1 + \sqrt{2})}$

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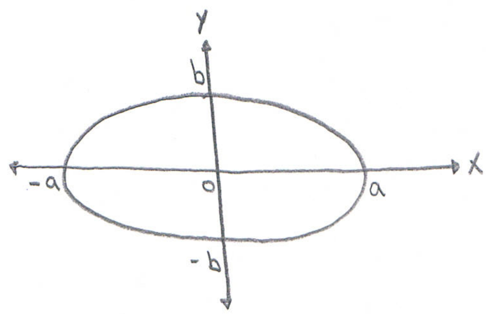
$f(x) = \cos x$

$f'(x) = -\sin x$

$1 + f'(x)^2 = 1 + \sin^2 x$

$$\boxed{L = \int_0^{2\pi} \sqrt{1 + \sin^2 x}\, dx}$$

18 FIRST, OBSERVE THAT THIS IS THE EQUATION OF AN ELLIPSE.



WE HAVE $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$y^2 = b^2\left(1 - \dfrac{x^2}{a^2}\right)$

$y = \pm b\sqrt{1 - \dfrac{x^2}{a^2}}$

LET'S JUST DEAL WITH $y > 0$

& DOUBLE OUR ANSWER AT THE END.

SO $f(x) = b\left(1 - a^{-2}x^2\right)^{1/2}$

$f'(x) = \frac{1}{2} b \left(1 - a^{-2}x^2\right)^{-1/2}\left(-2a^{-2}x\right)$

$1 + f'(x)^2 = 1 + \frac{1}{4} b^2 \left(\dfrac{1}{1 - \frac{x^2}{a^2}}\right)\left(4\dfrac{x^2}{a^4}\right) = 1 + \dfrac{b^2}{a^2}\left(\dfrac{x^2}{a^2 - x^2}\right)$

$$\therefore \boxed{L = 2\int_{-a}^{a} \sqrt{1 + \frac{b^2}{a^2}\left(\frac{x^2}{a^2 + x^2}\right)}\, dx}$$