§10.4 CROSS PRODUCT (VECTOR PRODUCT)

9/3/2014

Def: LET $\vec{a} = \langle a_1, a_2, a_3 \rangle$, $\vec{b} = \langle b_1, b_2, b_3 \rangle$.

THEN $\vec{a} \times \vec{b} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle$

NOTE: CROSS PRODUCT IS ONLY DEFINED FOR 3D VECTORS.

TRICK FOR REMEMBERING:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \hat{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \hat{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \hat{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

DETERMINANTS

e.g. COMPUTE $\langle 5, -8, 5 \rangle \times \langle -3, 1, -4 \rangle$

Thm: $\vec{a} \times \vec{b}$ (THE VECTOR) IS ORTHOGONAL TO BOTH $\vec{a}$ & $\vec{b}$.

Proof: $(\vec{a} \times \vec{b}) \cdot \vec{a} = \langle a_2 b_3 - a_3 b_2,\ a_3 b_1 - a_1 b_3,\ a_1 b_2 - a_2 b_1 \rangle \cdot \langle a_1, a_2, a_3 \rangle$

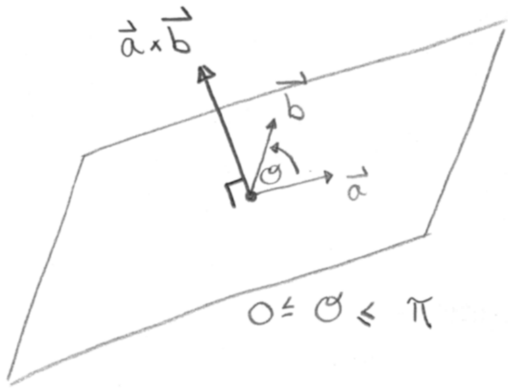
$$= a_1 a_2 b_3 - a_1 a_3 b_2 + a_2 a_3 b_1 - a_2 a_1 b_3 + a_3 a_1 b_2 - a_3 a_2 b_1$$

$= 0.$

& SIMILARLY, $(\vec{a} \times \vec{b}) \cdot \vec{b} = 0$. $\square$

DIRECTION: $\vec{a} \times \vec{b}$ POINTS PERPENDICULAR TO PLANE CONTAINING $\vec{a}, \vec{b}$

ACCORDING TO RIGHT HAND RULE



FINGERS CURL FROM $\vec{a}$ TO $\vec{b}$.
THUMB POINTS IN DIRECTION OF $\vec{a} \times \vec{b}$.

$\boxed{\text{MAGNITUDE: (THM) IF } \theta \text{ IS ANGLE BETWEEN } \vec{a} \text{ \& } \vec{b} \ (0 \le \theta \le \pi) \text{ THEN } |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta.}$

PROOF: $|\vec{a} \times \vec{b}|^2 = (a_2b_3 - a_3b_2)^2 + (a_3b_1 - a_1b_3)^2 + (a_1b_2 - a_2b_1)^2$

$$= (a_1^2 + a_2^2 + a_3^2)(b_1^2 + b_2^2 + b_3^2) - (a_1b_1 + a_2b_2 + a_3b_3)^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - (\vec{a} \cdot \vec{b})^2$$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{a}|^2|\vec{b}|^2 \cos^2\theta$$

$$= |\vec{a}|^2|\vec{b}|^2 (1 - \cos^2\theta)$$

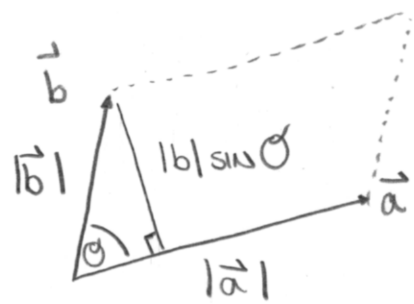
$$= |\vec{a}|^2|\vec{b}|^2 \sin^2\theta$$

$\Rightarrow |\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}||\sin\theta| = |\vec{a}||\vec{b}| \sin\theta$ SINCE $0 \le \theta \le \pi$ □

COR: IF $\vec{a}, \vec{b}$ NON ZERO, THEN $\vec{a} \parallel \vec{b} \iff \vec{a} \times \vec{b} = 0$.

PROOF: ...

FACT: $|\vec{a} \times \vec{b}|$ = AREA OF PARALLELOGRAM DETERMINED BY $\vec{a}$ & $\vec{b}$



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e.g. FIND A VECTOR $\perp$ TO PLANE THROUGH THE 3 POINTS

$$(1, 2, 1), (1, 4, 2), (3, -3, -2).$$

e.g. FIND AREA OF TRIANGLE WITH SAME VERTICES.

NOTE: $\hat{i} \times \hat{j} = \hat{k}$ | $\hat{j} \times \hat{k} = \hat{i}$ | $\hat{k} \times \hat{i} = \hat{j}$

$\hat{j} \times \hat{i} = -\hat{k}$ | $\hat{k} \times \hat{j} = -\hat{i}$ | $\hat{i} \times \hat{k} = -\hat{j}$

- NOT COMMUTATIVE!

NOTE: $\hat{i} \times (\hat{i} \times \hat{j}) = \hat{i} \times \hat{k} = -\hat{j}$ - NOT ASSOCIATIVE!

$(\hat{i} \times \hat{i}) \times \hat{j} = \vec{0} \times \hat{j} = \vec{0}$

PROPERTIES OF CROSS PRODUCT

LET $\vec{a}, \vec{b}, \vec{c}$ BE VECTORS & c BE A SCALAR.

(i) $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ [ANTI-COMMUTATIVE]

(ii) $(c\vec{a}) \times \vec{b} = \vec{a} \times (c\vec{b}) = c(\vec{a} \times \vec{b})$

(iii) $\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} \times \vec{b}) + (\vec{a} \times \vec{c})$ [LEFT-DISTRIBUTION]

(iv) $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$ [RIGHT-DISTRIBUTION]

(v) $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$

"SCALAR TRIPLE PRODUCT"

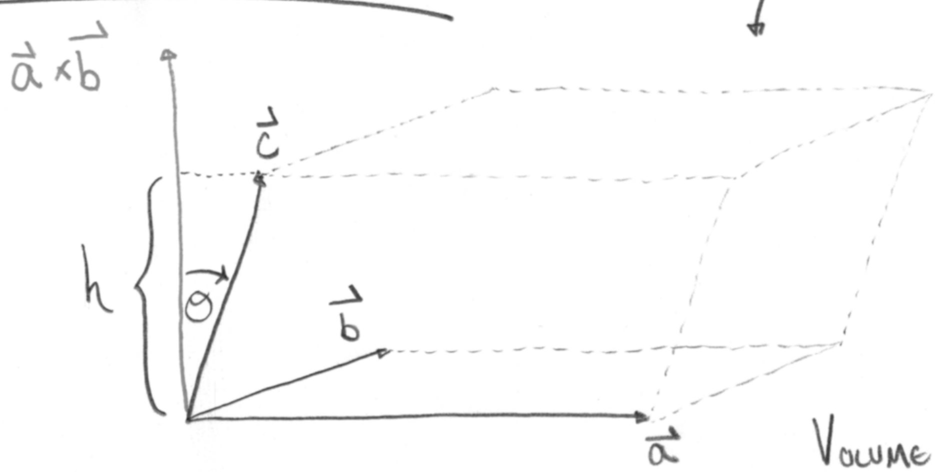
(vi) $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{a} \times \vec{c}) - \vec{c} \cdot (\vec{a} \times \vec{b})$

MNEMONIC: BAC − CAB

"VECTOR TRIPLE PRODUCT"

SCALAR TRIPLE PRODUCT

PARRALELOPIPED DETERMINED BY $\vec{a}, \vec{b}, \vec{c}$.



AREA OF BASE $B = |\vec{a} \times \vec{b}|$

HEIGHT $h = |\vec{c}||\cos\theta|$

VOLUME $V = Bh$

$= |\vec{a} \times \vec{b}||\vec{c}||\cos\theta|$

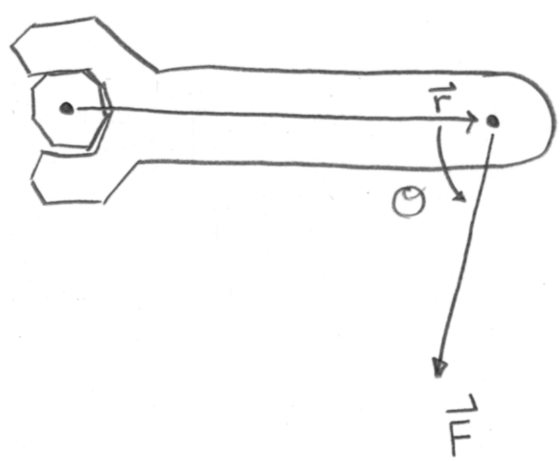
$= |(\vec{a} \times b) \cdot \vec{c}|$

NOTE: VECTORS $\vec{a}, \vec{b}, \vec{c}$ ARE COPLANAR

$\Longleftrightarrow$ THE VOLUME OF THEIR CORRESPONDING PARALLELOPIPED IS 0.

e.g. SHOW THAT $\langle -2, 1, -2 \rangle, \langle 7, -1, 8 \rangle, \langle 5, 0, 6 \rangle$ ARE COPLANAR.

TORQUE



$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}||\vec{F}| \sin \theta$$